Borsuk-Ulam type theorems and their discrete analogs

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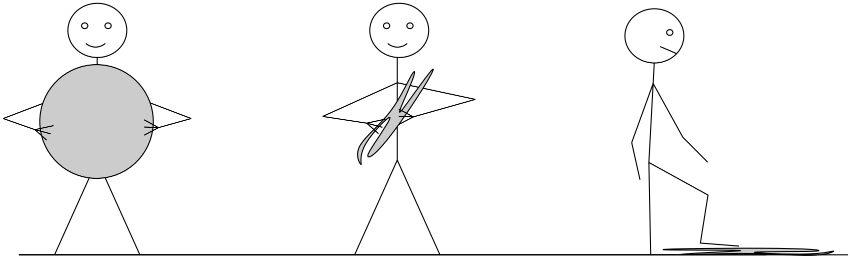
The Borsuk-Ulam theorem



papers

O. R. Musin, Borsuk–Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* **140** (2012)

O. R. Musin, Extensions of Sperner and Tucker's lemma for manifolds, *J. of Combinatorial Theory Series A*, **132** (2015)

O. R. Musin and A. Yu. Volovikov, Borsuk–Ulam type spaces, *Mosc. Math. J.*, **15:4** (2015)

O. R. Musin, Generalizations of Tucker–Fan–Shashkin lemmas, *Arnold Math J.*, **2:3** 2016

O. R. Musin and A. Yu. Volovikov, Tucker type lemmas for G-spaces, preprint, arXiv:1612.07314, 2021

A. V. Malyutin and O. R. Musin, Neighboring mapping points theorem, arXiv:1812.10895, 2021

papers

O. R. Musin, Extensions of Sperner and Tucker's lemma for manifolds, *J. of Combinatorial Theory Series A*, **132** (2015)

O. R. Musin, Sperner type lemma for quadrangulations, *Mosc. J. of Combinatorics and Number Theory*, **5** (2015).

O. R. Musin, Homotopy invariants of covers and KKM type lemmas, *Algebraic & Geometric Topology*, **16** (2016)

O. R. Musin, KKM type theorems with boundary conditions, *J. Fixed Point Theory Appl.*, **19** (2017)

O. R. Musin and Jie Wu, Cobordism classes of maps and covers for spheres, *Topology Appl.*, **237** (2018)

The Borsuk-Ulam theorem

The Borsuk - Ulam theorem (Borsuk, 1933). Four equivalent statements:
(a) For every continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.
(b) For every antipodal (i.e. $f(-x) = -f(x)$) continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = 0$.
(c) There is no antipodal continuous mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.
(d) There is no continuous mapping $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ that is antipodal on the boundary.

The Borsuk-Ulam theorem

Der Zweck dieser Arbeit ist, folgende drei Sätze zu beweisen:

Satz I [6]. *Jede antipodentreue Abbildung von S_n ist wesentlich.*

Satz II [7]. *Ist $f \in {R^n}^{S_n}$ (d. h. bildet f die Sphäre S_n auf einen Teil von R^n ab), so gibt es einen derartigen Punkt $p \in S_n$, dass $f(p) = f(p^*)$ ist.*

Satz III. *Sind $A_0, A_1, \ldots, A_n$ in sich kompakte Mengen von denen keine zwei antipodische Punkte der Sphäre S_n enthält, so enthält die Summe $\sum_{i=0}^{n} A_i$ die Sphäre S_n nicht.*

The Lyusternuk-Shnirelman theorem

Lyusternik and Shnirelman proved in 1930 that for any cover $F_1, \ldots, F_{n+1}$ of the sphere $\mathbb{S}^n$ by $n+1$ closed sets, there is at least one set containing a pair of antipodal points (that is, $F_i \cap (-F_i) \neq \emptyset$). Equivalently, for any cover $U_1, \ldots, U_{n+1}$ of $\mathbb{S}^n$ by $n+1$ open sets, there is at least one set containing a pair of antipodal points.

Tucker's lemma

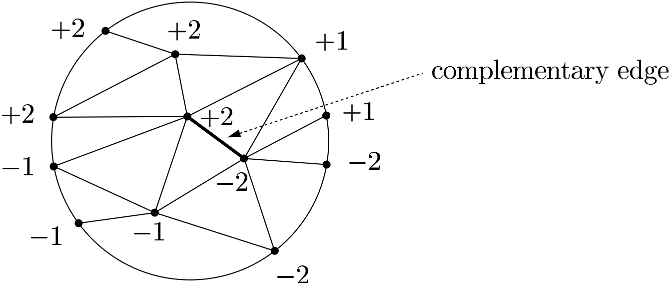
Theorem (Tucker, 1945)

Let Λ be a triangulation of the ball $\mathbb{B}^d$ that is antipodally symmetric on the boundary. Let

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \dots, +d, -d\}$$

be a labelling of the vertices of Λ that satisfies $L(-v) = -L(v)$ for every vertex v on the boundary $\mathbb{B}^d$. Then there exists an edge in Λ that is "complementary": i.e., its two vertices are labelled by opposite numbers.

Tucker's lemma



Tucker's lemma for spheres

Theorem

Let Λ be an antipodal triangulation of $\mathbb{S}^d$. Let

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +d, -d\}$$

be an antipodal labelling of the vertices of Λ that satisfies $L(-v) = -L(v)$ for all vertices. Then Λ contains a complimentary edge.

Fan's lemma

Theorem (Ky Fan, 1952)

Let Λ be an antipodal triangulation of $\mathbb{S}^d$. Suppose that each vertex v of Λ is assigned a label $L(v)$ from $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that $L(-v) = -L(v)$. Suppose this labelling does not have complementary edges. Then there are an odd number of d-simplices of Λ whose labels are of the form $\{k_0, -k_1, k_2, \ldots, (-1)^d k_d\}$, where $1 \leq k_0 < k_1 < \ldots < k_d \leq n$. In particular, $n \geq d + 1$.

P. Bacon, 1966

Theorem

Let X be a normal topological space with a free continuous involution $A : X \to X$. Then the following statements are equivalent:

1. (X, A) *is a BUT–space, i. e., for any continuous mapping* $f : X \to \mathbb{R}^n$ *there is* $x \in X$ *such that* $f(A(x) = f(x)$.
2. (X, A) *is a* LS_n*–space, i. e. for any cover* $C_1, \ldots, C_{n+1}$ *of* X *by* $n+1$ *closed (respectively, by* $n+1$ *open) sets, there is at least one set containing a pair* $(x, A(x))$.
3. (X, A) *is a* T_n*–space (Tucker space), i. e. for any covering of* X *by a family of* $2n$ *closed (respectively, of* $2n$ *open) sets* $\{C_1, C_{-1}, \ldots, C_n, C_{-n}\}$*, where* $C_{-i} = A(C_i)$*, for all* i*, there is* k *such that* C_k *and* C_{-k} *have a common intersection point.*

P. Bacon, 1966

4. (X, A) *is a* TB_n*–space (Tucker–Bacon space), i. e., if each of* $C_1, C_2, \ldots, C_{n+2}$ *is a closed subset of* X,

$$\bigcup_{i=1}^{n+2} C_i = X, \quad \bigcup_{i=1}^{n+2} (C_i \cap A(C_i)) = \emptyset,$$

then for any j *there is a point* p *in* X *such that*

$$p \in \bigcap_{i=1}^{j} C_i \;\; \textit{and} \;\; A(p) \in \bigcap_{i=j+1}^{n+2} C_i.$$

5. (X, A) *is an* Y_n*–space (Yang space).* Y_n *can be define recursively:* Y_0 *contains all* (X, A), $(X, A) \in Y_n$ *if a closed subset* F *in* X *is such that* $F \cup A(F) = X$, *then* $F \cap A(F)$ *is an* Y_{n-1}*–space.*

The Borsuk-Ulam theorem

One of the most interesting proofs of this theorem is Bárány's geometric proof:

I. Bárány, Borsuk's theorem through complementary pivoting, *Math. Programing*, **18** (1980), 84-88.

J. Matoušek, Using the Borsuk-Ulam theorem, Springer-Verlag, Berlin, 2003.

Bárány's proof

Let $X = \mathbb{S}^n \times [0,1]$, $X_0 = \mathbb{S}^n \times \{0\}$, and $X_1 = \mathbb{S}^n \times \{1\}$. Let $\tau(x,t) = (-x,t)$, where $(x,t) \in X$, $x \in \mathbb{S}^n$, and $t \in [0,1]$. Clearly, τ is a free involution on X.

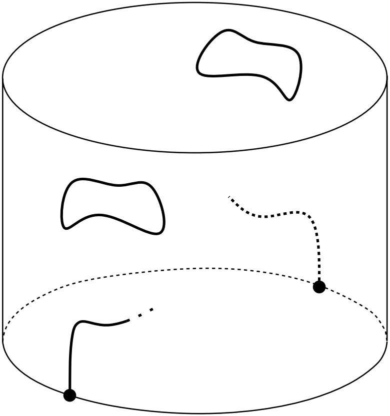
The first step of Bárány's proof is to show that any continuous antipodal (i.e. $F(\tau(x)) = -F(x)$) map $F : X \to \mathbb{R}^n$ can be approximated by "sufficiently generic" antipodal maps.
Let $f_i : \mathbb{S}^n \to \mathbb{R}^n$, where $i = 0, 1$, be antipodal generic maps. Let

$$F(x,t) = tf_1(x) + (1-t)f_0(x).$$

Bárány's proof

Since F is generic, the set $Z_F := F^{-1}(0)$ is a manifold of dimension one. Then Z_F consists of arcs $\{\gamma_k\}$ with ends in $Z_{f_i} := Z_F \bigcap X_i = f_i^{-1}(0)$ and cycles which do not intersect X_i. Note that $\tau(Z_F) = Z_F$ and $\tau(\gamma_i) = \gamma_j$ with $i \neq j$. Therefore, (Z_F, Z_{f_0}, Z_{f_1}) is a $\mathbb{Z}_2$-*cobordism*. It is not hard to see that Z_{f_0} *is* $\mathbb{Z}_2$-*cobordant to* Z_{f_1} if and only if $|Z_{f_1}| = |Z_{f_0}| = 4k + 2$ for some integer k.

Bárány's proof



Bárány's proof

To complete the proof, take f_0 as the standard orthogonal projection of $\mathbb{S}^n$ onto $\mathbb{R}^n$:

$$f_0(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n), \quad \text{where} \quad x_1^2 + \dots + x_{n+1}^2 = 1.$$

Since $|Z_{f_0}| = 2$, we have $|Z_{f_1}| = 4k + 2$ for some integer k. This equality shows that for any antipodal generic f_1 the set $Z_{f_1} = f_1^{-1}(0)$ is not empty.

Borsuk-Ulam theorem for the double torus

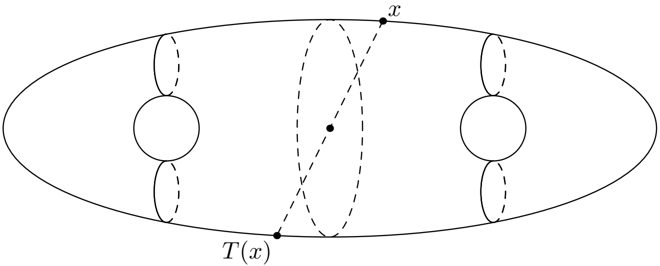


Figure: The double torus that is centrally symmetric embedded to $\mathbb{R}^3$.

Borsuk-Ulam theorem for the double torus

Theorem

Let M_2^2 denote the double torus that is centrally symmetric embedded to $\mathbb{R}^3$. Let $T(x) := -x,\ x \in M_2^2$.

(a) For every continuous mapping $f : M_2^2 \to \mathbb{R}^2$ there exists a point $x \in M_2^2$ with $f(x) = f(T(x))$.

(b) For every antipodal (i.e. $g(T(x)) = -g(x)$) continuous mapping $g : M_2^2 \to \mathbb{R}^2$ there exists a point $x \in M_2^2$ with $g(x) = 0$.

$\mathbb{Z}_2$-maps

Let us consider a closed smooth manifold M with a free smooth involution $T : M \to M$, i.e. $T^2(x) = x$ and $T(x) \neq x$ for all $x \in M$. For any $\mathbb{Z}_2$-manifold (M, T) we say that a map $f : M^m \to \mathbb{R}^n$ is *antipodal* (or equivariant) if $f(T(x)) = -f(x)$.

We say that a closed $\mathbb{Z}_2$-manifold (M, T) is a *BUT (Borsuk-Ulam Type) manifold* if for any continuous map $F : M^n \to \mathbb{R}^n$ there is a point $x \in M$ such that

$$F(T(x)) = F(x).$$

In other words, if a continuous map $f : M^n \to \mathbb{R}^n$ is antipodal, then the set $Z_f := f^{-1}(0)$ is not empty.

BUT manifolds

Theorem (M., 2012)

Let M^n be a closed connected manifold with a free involution T. Then the following statements are equivalent:

(a) For any antipodal continuous map $f : M^n \to \mathbb{R}^n$ the set Z_f is not empty.

(b) M admits an antipodal continuous transversal map $h : M^n \to \mathbb{R}^n$ with $|Z_h| = 4k+2,\ k \in \mathbb{Z}$.

(c) For any equivariant triangulation Λ of M and for any Tucker's labeling of $V(\Lambda)$ there is a complementary edge.

(d) $[M^n, T] = [\mathbb{S}^n, A] + [V^1][\mathbb{S}^{n-1}, A] + \ldots + [V^n][\mathbb{S}^0, A]$ in $\mathfrak{N}_n(\mathbb{Z}_2)$.

$\mathbb{Z}_2$-cobordisms.

We write $\mathfrak{N}_n$ for the group of unoriented cobordism classes of n-dimensional manifolds. Thom's cobordism theorem says that the graded ring of cobordism classes $\mathfrak{N}_*$ is $\mathbb{Z}_2[x_2, x_4, x_5, x_6, \ldots]$ with one generator x_k in each degree k not of the form $2^i - 1$. Note that $x_{2k} = [\mathbf{RP}^{2k}]$.

Let $\mathfrak{N}_*(\mathbb{Z}_2)$ denote the unoriented cobordism group of free involutions. Then $\mathfrak{N}_*(\mathbb{Z}_2)$ is a free $\mathfrak{N}_*$-module with basis $[\mathbb{S}^n, A]$, $n \geq 0$, where $[\mathbb{S}^n, A]$ is the cobordism class of the antipodal involution on the n-sphere. Thus, each $\mathbb{Z}_2$-manifold (M, T) in $\mathfrak{N}_n(\mathbb{Z}_2)$ can be uniquely represented in the form:

$$[M, T] = \sum_{k=0}^{n} [V^k][\mathbb{S}^{n-k}, A].$$

Shashkin lemma (1996)

Theorem

Let Θ be a triangulation of a planar polygon that antipodally symmetric on the boundary. Let

$$L : V(\Theta) \to \{+1, -1, +2, -2, +3, -3\}$$

be a labelling of the vertices of Θ that satisfies $L(-v) = -L(v)$ for every vertex v on the boundary. Suppose that this labelling does not have complementary edges. Then for any numbers a, b, c, where $|a| = 1$, $|b| = 2$, $|c| = 3$, the total number of triangles in Θ with labels (a, b, c) and $(-a, -b, -c)$ is odd.

Shashkin lemma for BUT–manifolds

$$\Pi_{d+1} := \{+1, -1, +2, -2, \ldots, +(d+1), -(d+1)\}$$

Theorem (M., 2016)

Let (M, T) be a d-dimensional BUT–manifold. Let Θ be an antipodally symmetric triangulation of M. Let $L : V(\Theta) \to \Pi_{d+1}$ be an antipodal labelling of Θ. Suppose that this labelling does not have complementary edges. Then for any set of labels $\Lambda := \{\ell_1, \ell_2, \ldots, \ell_{d+1}\} \subset \Pi_{d+1}$ with $|\ell_i| = i$ for all i, the number of d–simplices in Θ that are labelled by Λ is odd.

Topological index

Consider a group G as a discrete free G-space. Let $J^m(G) = G * \cdots * G$ be the join of m-copies of G with the diagonal action of G.

Let X be a free G-space. *Topological index* $\operatorname{t-ind}^{\mathrm{G}} X$ equals minimal n such that there exists an equivariant map $X \to J^{n+1}(G)$. If no such n exists, then $\operatorname{t-ind}^{\mathrm{G}} X = \infty$.

If $G = \mathbb{Z}_2$ then $J^{m+1}(\mathbb{Z}_2)$ is equivariantly homeomorphic to S^m, since $SY = Y * \mathbb{Z}_2$, where SY is the suspension, and

$$S^m = SS^{m-1} = S^{m-1} * \mathbb{Z}_2 = S^{m-2} * \mathbb{Z}_2 * \mathbb{Z}_2 = \cdots = J^{m+1}(\mathbb{Z}_2).$$

Tucker type lemmas for G-spaces

Let X be a G–simplicial complex, where G is a finite group. An *equivariant* (G,n)*-labeling (coloring)* of X is an equivariant map $V(X) \to C := G \times \{1,\dots,n\}$, where G acts on the first factor by left multiplication and on the second factor the action is trivial.

An edge in X is called *complementary* if labels of its vertices belong to the same orbit in C. For (G,n)-labeling it means that vertices of a complementary edge have the form (g_1,k) and (g_2,k), $g_1 \neq g_2$, for some $k \in \{1,\dots,n\}$.

Theorem (M. and A. Volovikov)

$\text{t-ind}^{\mathrm{G}} X \geq d$ *if and only if for any equivariant* (G,d)*-labeling of the vertex set of an arbitrary equivariant triangulation of* X *there exists a complementary edge.*

Cohomological index

Let X be a free G-space. We define $\mathrm{ind}^G X$, the integer cohomological index of X, as its Schwarz's homological genus minus 1.

We say that $h : X_0 \to X$ is *n-cohomological trivial* (n-c.t. map) over R if $h^* : H^n(X; R) \to H^n(X_0; R)$ is the trivial homomorphism of cohomology groups with coefficients in R in dimension n. In the case when h is an embedding we call X_0 an n-c.t.-subspace of X over R.

Tucker type lemmas for bounded spaces

Theorem (M. and A. Volovikov)

Assume that $\operatorname{ind}^{\mathrm{G}} X = n-1$ *and that* X_0 *is an* $(n-1)$*-c.t.-subspace of* X *over* $\mathbb{Z}$*. Then for any* (G,n)*-labeling of the vertex set of an arbitrary triangulation of* X *which is equivariant on* X_0 *there exists a complementary edge.*

As a partial case we obtain:

Theorem (M. and A. Volovikov)

Let M^n *be a compact PL manifold with boundary. Suppose that* ∂M *is homeomorphic to the sphere* $\mathbb{S}^{n-1}$ *and there exists a free PL action of a group* G *on* $\partial M \approx \mathbb{S}^{n-1}$*. Then for any* (G,n)*-labeling of the vertex set of an arbitrary triangulation of* M *that is an equivariant on the boundary there exists a complementary edge.*

Knot Theory

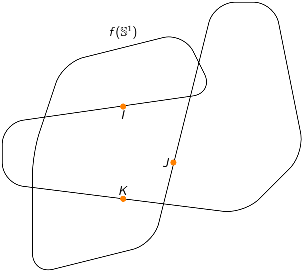
A. V. Malyutin, *On the question of genericity of hyperbolic knots,* Int. Math. Res. Not. (2018)

We say that two arcs of a knot diagram D are *neighboring* if they are contained in the boundary of the same region.
Denote by $\rho(I, J)$ the minimal number of consecutive arcs between I and J.

Lemma

Any regular knot projection with $n > 0$ double points has a pair of neighboring arcs I and J with $\rho(I, J) \geq 2n/3$.

Knot Theory



$2n = 10$
$\rho(I, J) = 5$
$\rho(I, K) = 2$
$\rho(J, K) = 3$

Lemma

Any regular knot projection with $n > 0$ double points has a pair of neighboring arcs I and J with $\rho(I, J) \geq 2n/3$.

The lemma can be proved via the Sperner Lemma or KKM (Knaster–Kuratowski–Mazurkiewicz) Lemma.

f–neighbors

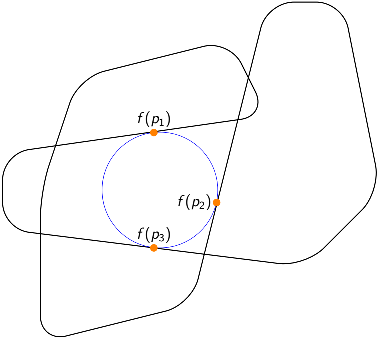
Let $f : \mathbb{S}^m \to \mathbb{R}^n$ be a smooth map. We say that two points a and b in $\mathbb{S}^m$ are *topological f–neighbors* if $f(a)$ and $f(b)$ can be connected by a continuous path in $\mathbb{R}^n$, whose interior does not meet $f(\mathbb{S}^m)$. Let a and b be topological f–neighbors in $\mathbb{S}^m$.

1. if $m = n$ then $f(a) = f(b)$,
2. if $m = 1, n = 2$ then $f(a)$ and $f(b)$ belong to the boundary of the same connected component of $\mathbb{R}^2 \setminus f(\mathbb{S}^1)$,
3. if $n \geq m + 2$ then (a, b) can be any pair of points in $\mathbb{S}^m$.

We say that a and b in $\mathbb{S}^m$ are *visual f–neighbors* if the interior of the line segment in $\mathbb{R}^n$ with endpoints at $f(a)$ and $f(b)$ does not intersect $f(\mathbb{S}^m)$.

Spherical f–neighbors

Let $f : X \to Y$ be a continuous map. Points $\{p_i\}$ are *f–neighbors* if there exists a sphere S_R of radius R in Y such that $\{f(p_i)\}$ lie on S_R and there are no points of $f(X)$ inside of S_R.



f–neighbors Theorem 1

Theorem

Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f : \mathbb{S}^m \to \mathbb{R}^n$ be a continuous map. Then there are points p and q in $\mathbb{S}^m$ such that

- $\|p - q\| \geq \sqrt{2 \cdot \frac{m+2}{m+1}}$;
- *$f(p)$ and $f(q)$ lie on the boundary ∂B of a closed metric ball $B \subset \mathbb{R}^n$ whose interior does not meet $f(\mathbb{S}^m)$. In other words, p and q are (spherical) f–neighbors.*

f–neighbors Theorem 2

Theorem (is equivalent to the BUT) *Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f : \mathbb{S}^m \to \mathbb{R}^m$ be a continuous map. Then each point inside of $\mathbb{S}^m$ is contained in a straight line segment $[a, b]$ with $f(a) = f(b)$.*

Theorem (2)

Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f : \mathbb{S}^m \to \mathbb{R}^n$ be a continuous map. Then each point inside of $\mathbb{S}^m$ is contained in the convex hull of a family of spherical f–neighbors.

Theorem 1 follows from Theorem 2 by the Jung theorem.

f–neighbors Theorem 3

Theorem (3)

Let Q be a compact subset in $\mathbb{R}^m$, let ∂Q be the boundary of Q, and let $f : \partial Q \to \mathbb{R}^n$ be a continuous map. Then every point of Q is contained in the convex hull of a family of spherical f–neighbors.

Delaunay maps

Let K be an abstract simplicial complex and let $f\colon K \to \mathbb{R}^m$ be a map. We say that f is a *Delaunay map* if $f(\Delta)$ is a simplex of $DT(f(K))$ for each simplex Δ of K.

In other words, f is Delaunay if it is a simplicial map from $|K|$ to the Delaunay triangulation of $f(K)$.

f–neighbors theorem for Delaunay maps

Theorem

Let V be the set of vertices of a (not necessarily convex) simplicial n-polytope M in $\mathbb{R}^n$, and let $f : V \to \mathbb{R}^m$ be a Delaunay map. Then for each point $p \in M$ there exist a collection $Z \subset V$ of f–neighbors such that the convex hull of Z contains p.

The theorem follows from the empty sphere property of Delaunay triangulations plus the Quillen's fiber lemma (or, alternatively, one can use Smale's homotopy version of Vietoris–Begle mapping theorem).

Delaunay approximation

Theorem

For any continuous map f of a compact simplicial space to $\mathbb{R}^m$ and for any $\epsilon > 0$, there exists an ϵ-approximation of f by a Delaunay map.

Delaunay approximation theorem + f–neighbors theorem for Delaunay maps $\Rightarrow$ Theorem 3 $\Rightarrow$ Theorem 2 $\Rightarrow$ Theorem 1

non–null–homotopic covers

$\mathcal{U} = \{U_1, \dots, U_n\}$ — an open cover of a normal topological space X
$\Phi = \{\varphi_1, \dots, \varphi_n\}$ — a partition of unity subordinate to $\mathcal{U}$
$v_1, \dots, v_n$ — the vertices of Δ^{n-1}

Set $h_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i$

Suppose $\bigcap_{i=1}^n U_i = \emptyset$. Then $h_{\mathcal{U},\Phi}$ is a continuous map $X \to S^{n-2}$.
The homotopy class $[h_{\mathcal{U},\Phi}]$ in $[X, S^{n-2}]$ does not depend on Φ.
We denote this class in $[X, S^{n-2}]$ by $[\mathcal{U}]$.

We say that an open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of X is *non–null–homotopic* if the intersection $\bigcap_{i=1}^n U_i$ is empty and $[\mathcal{U}] \neq 0$ in $[X, S^{n-2}]$.
The homotopy classes of covers are also well defined for closed sets.

Covering neighboring points theorem

Theorem

Let X be a normal topological space and M be a contractible metric space. Let $C := \{C_1, \dots, C_m\}$ be a non–null–homotopic closed cover of X. Then for every continuous map $f : X \to M$ there exist (not necessarily distinct) points $p_1, \dots, p_m$ with $p_i \in C_i$ for all $i = 1, \dots, m$ such that they are f–neighbors.

Corollary (cf. Theorem 1)

Let $\mathbb{S}^m$ be a unit sphere in $\mathbb{R}^{m+1}$ and let $f : \mathbb{S}^m \to M$ be a continuous map to a contractible metric space M. Then there are spherical f–neighbors p and q in $\mathbb{S}^m$ with

$$\|p - q\| \geq \sqrt{\frac{m+2}{m}}.$$

$\mathbb{R}^n \to$ contractible metric space

$\sqrt{2 \cdot \frac{m+2}{m+1}} \to \sqrt{\frac{m+2}{m}}$

Thank you