On topological classification of the structural stable dynamical systems

O. Pochinka, Nizhny Novgorod, Russia

Higher School of Economics

Rough dynamical systems

Concept roughness of a dynamical system was born in Nizhny Novgorod in 1937 (then Gorky). A. Andronov and L. Pontryagin considered a dynamical system

$$\dot{x} = v(x),$$

where v is a C^1-vector field on the plane, $x \in \mathbb{R}^2$ and suggest to call it rough if for any sufficiently small perturbation in the C^1-metric, there exists a homeomorphism close to the identity map which transforms the orbits of the original dynamical system to the orbits of the perturbed system (perturbed system is topologically equivalent to original one by a conjugating homeomorphism).

Criteria of the roughness

In the paper “A. Andronov and L. Pontryagin. Rough systems. Doklady Akademii Nauk SSSR. 1937. 14 (5): 247–250” for a dynamical system

$$\dot{x} = v(x),$$

where v is a C^1-vector field given on the unit disk and transversal to the boundary, was done following criteria for its roughness:

- number of the equilibrium points and periodic orbits is finite and they are hyperbolic;
- there are no saddle connections.

Leontovich-Mayer scheme

The topological classification (division into classes with respect to the topological equivalence) of structurally stable flows (dynamical systems with continuous time) on a bounded part of the plane and on the 2-sphere follows from the results by E. Leontovich-Andronova and A. Mayer. In the papers “E. Leontovich, A. Mayer. On trajectories defining qualitative structure of decomposition of the sphere into trajectories. Dokl. Akad. Nauk SSSR. 1937. 14 (5), 251–257” and “E. Leontovich, A. Mayer. On a scheme defining topological structure of decomposition into trajectories. Dokl. Akad. Nauk SSSR. 1955. 103 (4), 557–560” actually more general class of dynamical systems was considered. The classification was based on the ideas of Poincare-Bendixson to pick a set of specially chosen trajectories so that their relative position (Leontovich-Mayer scheme) fully define the qualitative structure of the decomposition of the phase space of the dynamical system into the trajectories.

Transition to a surface with positive genus

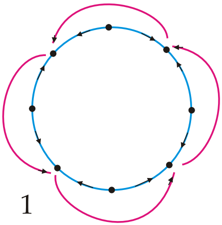
The principal difficulty in generalization of this result in case of arbitrary orientable surfaces of positive genus is the possibility of new types of motion — non-closed recurrent trajectories. The absence of such trajectories for structurally stable flows without singularities on the 2-torus at first was proved by A. Mayer. Actually in the paper "Mayer A.G. Rough transformation of the circle to the circle. Uch. Zap. GGU. 1939. Gorky, Pub. GGU, 12, 215-229." he introduced the rough notion for cascades (discrete dynamical systems), found the conditions of the roughness for cascades on the circle and also got the topological classification for these cascades.

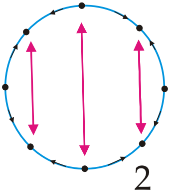
Rough transformations of circle

Let $R(\mathbb{S}^1)$ be class of rough transformations of the circle which consists of two subclasses $R_+(\mathbb{S}^1)$ and $R_-(\mathbb{S}^1)$ of preserving orientation and reverse orientation diffeomorphisms, accordingly.

1. For each diffeomorphism $\varphi \in R_+(\mathbb{S}^1)$ the non-wandering set $NW(\varphi)$ consists of $2n, n \in \mathbb{N}$ periodic orbits, each of them has period k.

2. For each diffeomorphism $\varphi \in R_-(\mathbb{S}^1)$ the non-wandering set $NW(\varphi)$ consists of $2q, q \in \mathbb{N}$ periodic points, two of them are fixed, others have period 2.



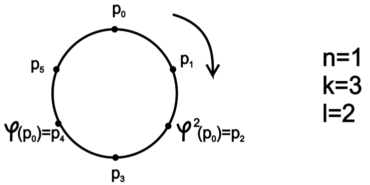


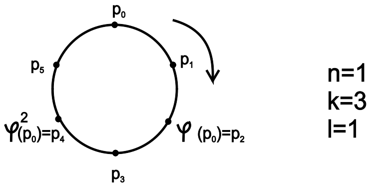
The preserving orientation case

Let $\varphi \in R_+(\mathbb{S}^1)$. Enumerate the periodic points from $NW(\varphi)$: $p_0, p_1, \ldots, p_{2nk-1}, p_{2nk} = p_0$ starting from arbitrary periodic point p_0 clockwise, then $\varphi(p_0) = p_{2nl}$ and (k, l) are coprime.

Two diffeomorphisms $\varphi; \varphi' \in R_+(\mathbb{S}^1)$ with parameters $n, k, l; n', k', l'$ are topologically conjugated if and only if $n = n', k = k'$ and at least one of the following assertions holds:

- $l = l'$,
- $l = k' - l'$.

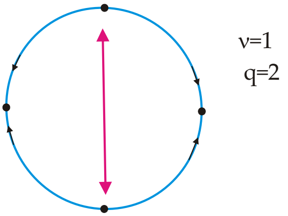


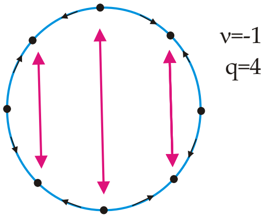


The reversing orientation case

For $\varphi \in R_-(\mathbb{S}^1)$ we set $\nu = -1;\ \nu = 0; \nu = +1$ if its fixed point are sources; sink and source; sinks, accordingly. Notice that $\nu = 0$ if q is odd and $\nu = \pm 1$ if q is even.

Two diffeomorphisms $\varphi; \varphi' \in R_-(\mathbb{S}^1)$ with parameters $q, \nu;\ q', \nu'$ are topologically conjugated if and only if $q = q'$ and $\nu = \nu'$.





Structural stability

In 1959 M. Peixoto introduced the concept of structural stability of flows to generalize the concept of roughness.
A flow f^t is called structurally stable if, for any sufficiently close flow g^t, there exists a homeomorphism h sending trajectories of the system g^t to trajectories of the system f^t. The original definition of a rough flow involved the additional requirement that the homeomorphism h be C^0-close to the identity map.
Peixoto proved that the concepts of roughness and structural stability for flows on 2-sphere are equivalent. In 1962 Peixoto proved that the conditions 1),2) above plus condition
3) all ω- and α-limit sets are contained in the union of the equilibrium points and the limit cycles
are necessary and sufficient for the structural stability of a flow on arbitrary orientable closed (compact and without boundary) surface and showed that such flows are dense in the space of all C^1-flows.

Morse-Smale systems

An immediate generalization of properties of rough flows on orientable surfaces leads to Morse-Smale systems (continuous and discrete). The non-wandering set of such a system consists of finitely many fixed points and periodic orbits, each of which is hyperbolic and the stable and unstable manifolds W_p^s and W_q^u intersect transversally for any distinct non-wandering points p, q.

Morse-Smale systems are named in 1960 after paper "Morse inequality for Dynamical Systems" Bull. Amer. Math. Soc. 1960, No. 66, 46-49" by S. Smale, where he introduced flows with the above properties (on manifolds of dimension greater than 2) and proved that they satisfy inequalities similar to the Morse inequalities.

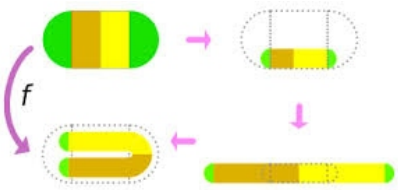
Citation

"We remark that systems satisfying 1)-3) may be very important because of the following possibilities.
(A) It seems at least plausible that system satisfying 1)-3) form an open dense set in the space (with the C^1-topology) of all vector fields on M^n.
(B) It seems likely that conditions 1)-3) are necessary and sufficient for X to be structurally stable in the sense of A. Andronov and L. Pontryagin (1937).
(A) and (B) have been provide for the case M^n is a 2-disk."

S. Smale

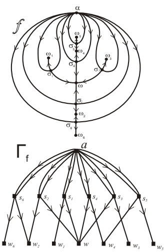
Morse-Smale systems do not exhaust the class of all rough systems

Later 1969 S. Smale and J. Palis showed that Morse-Smale systems are structurally stable. However, already in 1961 Smale proved that such systems do not exhaust the class of all rough systems via constructing a structurally stable diffeomorphism on the two-dimensional sphere $\mathbb{S}^2$ with infinitely many periodic points. This diffeomorphism is known now as the Smale's horseshoe.



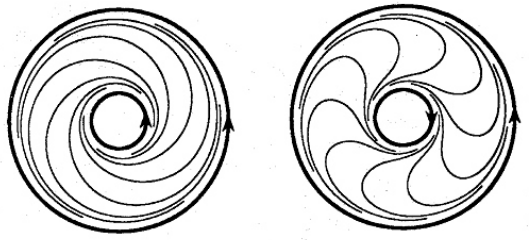
Peixoto's graph

In 1971 M. Peixoto generalized the Leontovich-Mayer's scheme for Morse-Smale flow on arbitrary surface as the directed graph whose vertices are in a one-to-one correspondence with fixed points and closed trajectories of the flow, and whose edges correspond to the connected components of the invariant manifolds of fixed points and closed trajectories. He proved that the isomorphism class of such directed graph is the complete topological invariant for the class of Morse-Smale systems on surfaces (where the isomorphisms preserve specially chosen subgraphs).



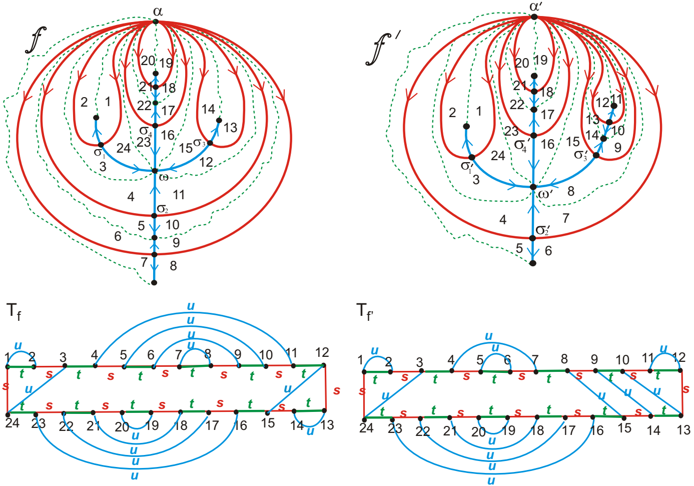
Oshemkov-Sharko approach

A. Oshemkov and V. Sharko in 1998 pointed out a certain inaccuracy concerning the Peixoto invariant due to the fact that an isomorphism of graphs does not distinguish between types of decompositions into trajectories for a domain bounded by two periodic orbits.

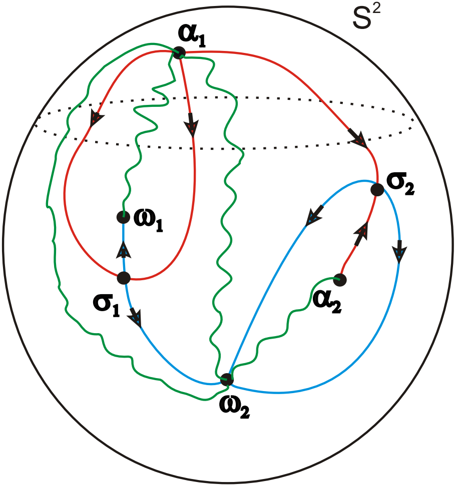


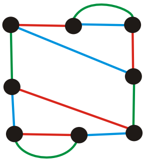
Three-colour graph

A. Oshemkov and V. Sharko suggest to use a three-colour graph.

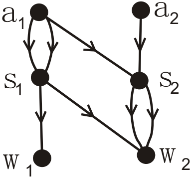


The graphs





Three-colored graph



Directed graph

Gradient-like difeomorphisms

Morse-Smale diffeomorphisms is called gradient-like if it has no heteroclinic points.

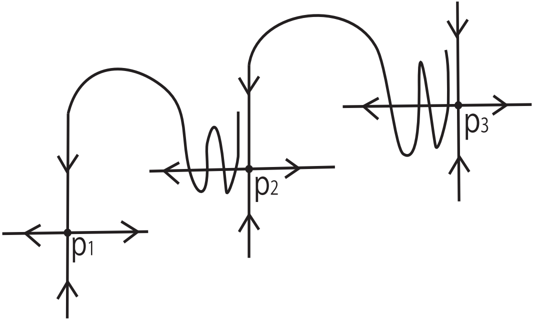


Figure: Heteroclinic points

Ch. Bonatti and R. Langevin in 1998 presented topological classification of arbitrary structurally stable diffeomorphisms of orientable surfaces using Markov partitions as complete invariant.

On topological conjugacy

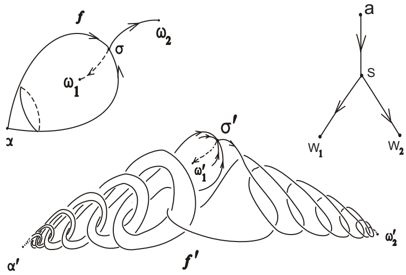
Let $f^t : M^n \to M^n$ ($f : M^n \to M^n$) be a flow or gradient-like diffeomorphism.

Is a graph (like to directed or three-colour) complete topological invariant?

- It is true for flows and $n = 2$ (A. Andronov, E. Leontovich-Andronova, A. Mayer (1937,1955) for sphere, M. Peixoto, 1971-1973 for any surfaces).
- It is true for diffeomorphisms and $n = 2$ (A. Bezdenezhnych, V. Grines 1985 V. Grines, S. Zinina, O. Pochinka 2014).
- It is true for flows and $n = 3$ (G. Fleitas 1975, Ya. Umanskii 1990).
- It is true for flows and diffeomorphisms on n-sphere, $n > 3$ withou heteroclinic intersection (S. Pilyugin 1978 for flows, V. Grines, E. Gurevich, V. Medvedev, O. Pochinka 2008–2015 for diffeomorphisms).

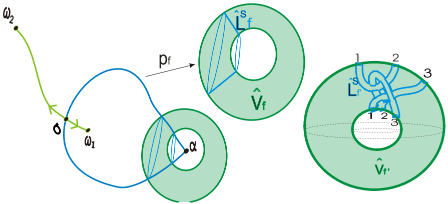
Combinatorial invariants

For wide class of Morse-Smale systems a graph is complete invariant (similar to Leontovich-Andronova and Mayer's scheme or Peixoto's graph for flows). Topological classification of even the simplest examples of Morse-Smale diffeomorphisms on 3-manifolds do not fit into the concept of selecting of the frame of the invariant manifolds of fixed points and periodic orbits. The reason for this surprising effect is the possibility of "wild" behavior of the separatrices of the saddle points. First diffeomorphism with wild separatrices was constructed by D. Pixton in 1977.



Classification of the Pixton's class $\mathcal{P}$

Let $f \in \mathcal{P}$. Set $V_f = W^u_\alpha \setminus \alpha$, $\hat{V}_f = V_f/_f$. Denote $p_f : V_f \to \hat{V}_f$ the natural projection. Then $\hat{V}_f$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, p_f is cover and $\hat{L}^s_f = p_f(W^s_\sigma \setminus \sigma)$ is homeomorphic to $\mathbb{T}^2$.



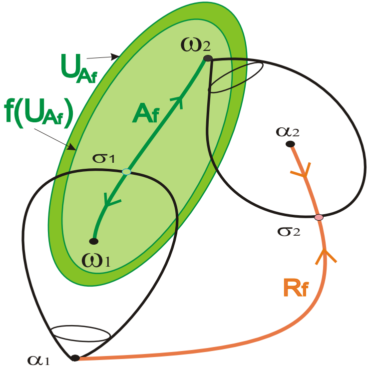
Theorem

(Ch. Bonatti, V. Grines, 2000) Diffeomorphisms $f, f' \in \mathcal{P}$ are topologically conjugated if and only if there is a homeomorphism $\hat{\varphi} : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$.

Global dynamic of Morse-Smale diffeomorphisms.

Let f be a Morse-Smale diffeomorphism on 3-manifold. Let us denote by $\Omega_q,\ q=0,1,2,3$ the set of all periodic points with Morse index q.

We set $A_f=\Omega_0\cup W^u_{\Sigma_1}$, $R_f=\Omega_3\cup W^s_{\Sigma_2}$. It is possible to prove that A_f (R_f) is a connected set which is an attractor (a repeller) of f.

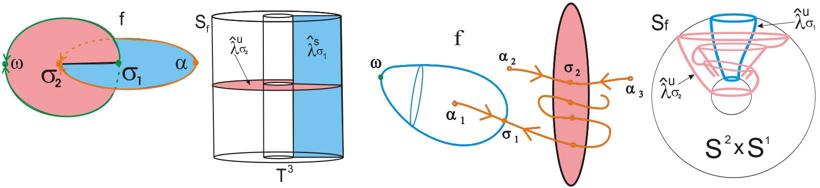


Scheme of Morse-Smale diffeomorphism f

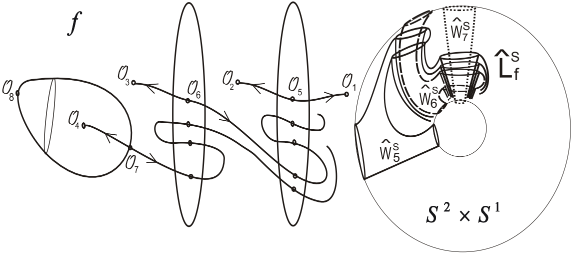
Set $V_f = M^3 \setminus (A_f \cup R_f)$ and $\hat{V}_f = V_f/f$. Denote by $p_f : V_f \to \hat{V}_f$ the natural projection and by $\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$ epimorphism, induced by cover p_f. Set $\hat{L}^s_f = p_f(W^s_{\Omega_1})$ and $\hat{L}^u_f = p_f(W^u_{\Omega_2})$.

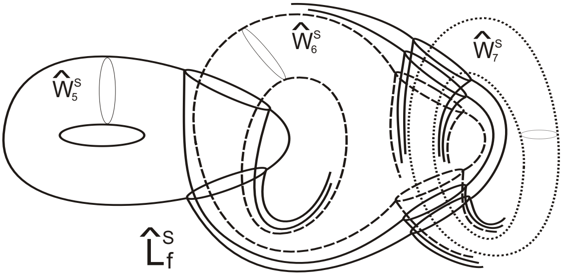
Definition

The collection $S_f = (\hat{V}_f, \eta_f, \hat{L}^u_f, \hat{L}^s_f)$ *is called scheme of the diffeomorphism* f.



Heteroclinic lamination





The results was obtained in collaboration with

Figure: Cr. Bonatti

Figure: V. Medvedev

Figure: E. Pecou

The scheme is complete invariant

Definition

The schemes $S_f = (\hat{V}_f, \eta_{_f}, \hat{L}^u_f, \hat{L}^s_f)$ and $S_{f'} = (\hat{V}_{f'}, \eta_{_{f'}}, \hat{L}^u_{f'}, \hat{L}^s_{f'})$ of diffeomorphisms f, f' are called equivalent if there is a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that

1) $\eta_{_f} = \eta_{_{f'}} \hat{\varphi}_*$;

2) $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$, $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$.

Theorem

(Ch. Bonatti, V. Grines, O. Pochinka, 2019) The diffeomorphisms f, f' are topologically conjugated if and only if their schemes are equivalent.

Hyperbolic non-wandering set

Definition

Nonwandering set $\Omega(f)$ is hyperbolic *if there is continuous df-invariant splitting*

$$T_{\Omega(f)}M^n = E^s_{\Omega(f)} \oplus E^u_{\Omega(f)}$$

of tangent subbundle $T_{\Omega(f)}M^n$ in sum of stable and unstable subbundels such that the following estimates hold:

$$\|df^k(v)\| \leq C\mathcal{B}^k\|v\|, \quad \|df^{-k}(w)\| \leq C\mathcal{B}^k\|w\|$$

for some real numbers $C > 0$ and $0 < \mathcal{B} < 1$,
and for any $v \in E^s_{\Omega(f)}, w \in E^u_{\Omega(f)}, k \in \mathbb{N}$.

A-diffeomorphisms

Definition

Diffeomorphism $f : M \to M$ *is called* A*-diffeomorphism if* f *satisfies to S. Smale axiom* A*, that is*

- *nonwandering set* $\Omega(f)$ *is hyperbolic;*
- *set of periodic points is dense in* $\Omega(f)$*.*

Axiom A and the strong condition of transversality are necessary and sufficient condition for the structural stability of a diffeomorphism $f : M^n \to M^n$.

Basic sets

According to S. Smale spectral theorem nonwandering set $\Omega(f)$ is the union of pair disjoint closed invariant sets each of which contains dense orbit under action of diffeomorphism f.

$$\Omega(f) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_l$$

where $l \geq 1$
An invariant set $\mathcal{B}$ of diffeomorphism $f : M \to M$ is called attractor if there is a closed neighborhood U of the set $\mathcal{B}$ such that $f(U) \subset int\ U$, $\bigcap\limits_{j \geq 0} f^j(U) = \mathcal{B}$. An invariant set is called repeller if it is attractor for f^{-1}.

Codimension one basic set is attractor or repeller

According to R. Plykin any basic set $\mathcal{B}$ of A-diffeomorphisms $f : M^n \to M^n$ such that $dim\ \mathcal{B} = n - 1$ is attractor or repeller. If $\mathcal{B}$ is an attractor then for any point $x \in \mathcal{B}$ unstable manifold $W^u(x)$ belongs to $\mathcal{B}$.

Definition

An attractor $\mathcal{B}$ of f is called expanding attractor of f if it's topological dimension is equal to dimension of $W^u(x)$ for any point $x \in \mathcal{B}$. A repeller $\mathcal{B}$ is called attracting repeller if it is expanding attractor of f^{-1}.

According to R. Plykin any expanding attractor of codimension one of $f : M^n \to M^n$ is locally homeomorphic to product of $(n - 1)$-disk and Cantor set.

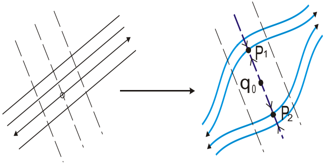
$f : M^2 \to M^2$, $dim\ \mathcal{B} = 2$

- $dim\ \mathcal{B} = 2$. Anosov diffeomorphism given on torus $\mathbb{T}^2$, first example belongs to R. Thom: $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

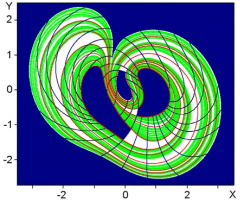
If $dim\ \mathcal{B} = 2$ then f is Anosov diffeomorphism and topological classification was first obtained by Ya.G. Sinai in 1968.

$f : M^2 \to M^2$, $dim\ \mathcal{B} = 1$

Any one-dimensional basic set of A-diffeomorphism is either expanding attractor or attracting repeller.
The first well known example is one-dimensional orientable attractor of DA-diffeomorphism given on torus $\mathbb{T}^2$.

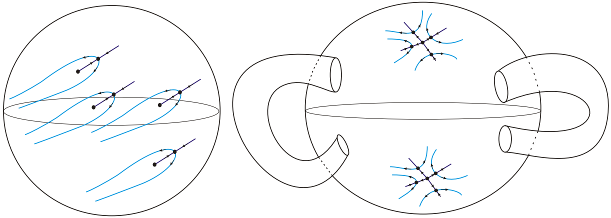


The second is celebrated non-orientable Plykin attractor of A-diffeomorphism given on S^2.



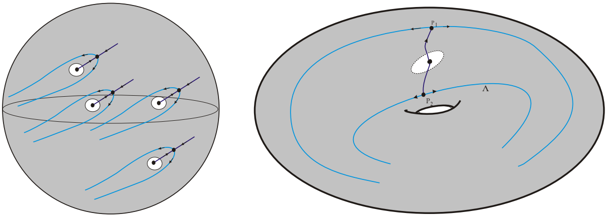
Classification of nontrivial attractors and repellers on orientable surfaces

The problem of topological classification of such basic sets was posed by L.P. Shilnikov in 1970 and was solved by V.Z. Grines in 1974 for orientable basic sets. Then this problem was solved completely for arbitrary attractors on surfaces in series papers by R.V. Plykin, A.Yu. Zhirov, X.X. Kalay, V. Grines.



Canonical support

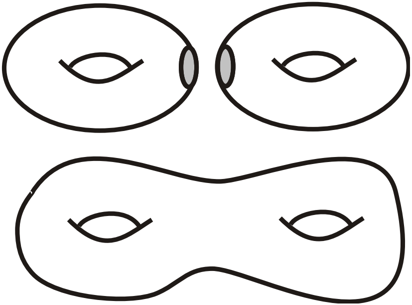
For arbitrary one-dimensional attractor $\mathcal{B} \subset \Omega(f)$ of diffeomorphism $f : M^2 \to M^2$ there is a compact orientable surface $N_{\mathcal{B}}$ — (canonical support) of negative Eiler characteristic $\chi(N_{\mathcal{B}})$ and diffeomorphism $f_{\mathcal{B}} = f|_{N_{\mathcal{B}}} : N_{\mathcal{B}} \to N_{\mathcal{B}}$.



Universal cover

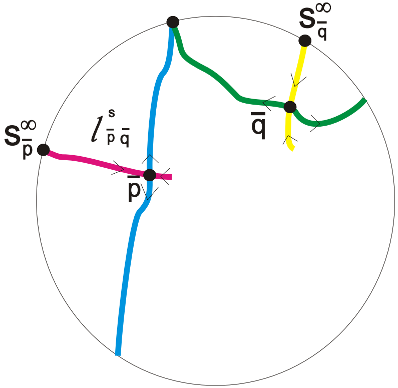
We consider the Poincare disk model of the hyperbolic plane as the unit open ball $\mathbb{U} = \{z \in \mathbb{C} \; : \; |z| < 1\}$ of the complex plane. The boundary of the ball $\mathbb{U}$ is called the *absolute of the hyperbolic plane* denoted by $\mathbb{E}$ ($\mathbb{E} = \partial\mathbb{U} = \{z \in \mathbb{C} \; : \; |z| = 1\}$).

If we glue two copies of $N_{\mathcal{B}}$ along the boundary components we get a surface F without boundary. Then there is a group $\mathbb{G}_{N_{\mathcal{B}}}$ of isometries and a connected set $\mathbb{U}_{N_{\mathcal{B}}} \subset \mathbb{U}$ such that $p_{\mathcal{B}} : \mathbb{U}_{N_{\mathcal{B}}} \to N_{\mathcal{B}}$ is a universal cover.



Asymptotic direction

Set $p_{\mathcal{B}}^{-1}(\mathcal{B}) = \bar{\mathcal{B}}$. Each curve $w_{\bar{x}}^{\delta\nu}$ has the asymptotic direction $\delta_{\bar{x}}^{\nu}$ for $t \to \nu\infty,\ \nu \in \{-,+\}$, that is if $cl(w_{\bar{x}}^{\delta\nu}) \setminus w_{\bar{x}}^{\delta\nu}$ consists of the point $\bar{x}$ and the point $\delta_{\bar{x}}^{\nu}$ which belongs to $\mathbb{E}$.

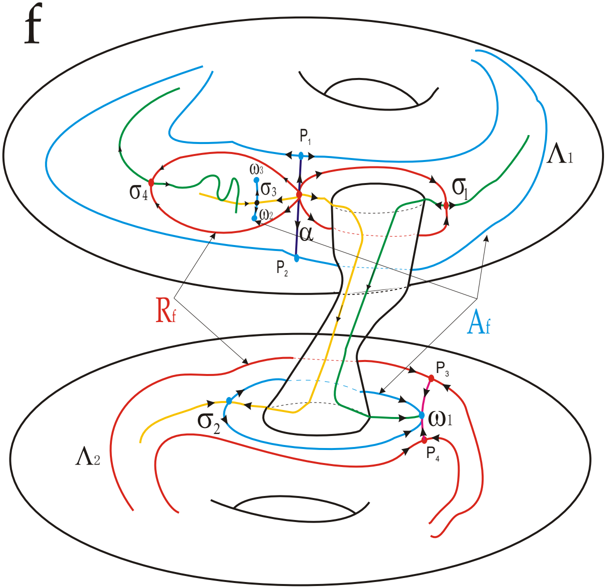


The classification theorem

A lift $\bar{f}_{\mathcal{B}} : \mathbb{U}_{N_{\mathcal{B}}} \to \mathbb{U}_{N_{\mathcal{B}}}$ of $f_{\mathcal{B}} = f|_{N_{\mathcal{B}}}$ with respect $p_{_{\mathcal{B}}}$ induces an automorphism $T_{\bar{f}_{\mathcal{B}}}$ of the group $\mathbb{G}_{N_{\mathcal{B}}}$ which acts by the formula $T_{\bar{f}_{\mathcal{B}}}(g) = \bar{f}_{\mathcal{B}} g \bar{f}_{\mathcal{B}}^{-1}$.

Theorem

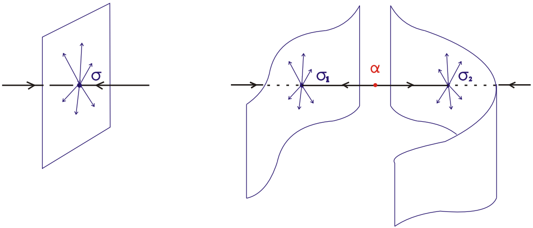
(V. Grines) Let $f : M^2 \to M^2$, $f' : M'^2 \to M'^2$ be A-diffeomorphisms with one-dimensional attractors $\mathcal{B}, \mathcal{B}'$ accordingly. Then $f|_{\mathcal{B}}$, $f'|_{\mathcal{B}'}$ are topologically conjugated if and only if there is an isomorphism $\psi_{_{\mathcal{B}}} : \mathbb{G}_{N_{\mathcal{B}}} \to \mathbb{G}_{N_{\mathcal{B}'}}$ such that $T_{\bar{f}'_{\mathcal{B}'}} = \psi_{_{\mathcal{B}}} T_{\bar{f}_{\mathcal{B}}} \psi_{_{\mathcal{B}}}^{-1}$ for some $\bar{f}_{\mathcal{B}}, \bar{f}'_{\mathcal{B}'}$.



f
P1
ω3
σ3
ω2
σ4
σ1
α
P2
Λ1
Rf
Af
P3
σ2
ω1
P4
Λ2

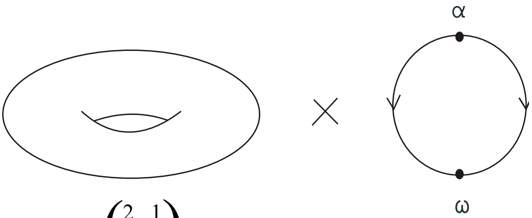
$n = 3$, M^3– closed orientable 3-manifold. Examples of two-dimensional expanding attractor

DA-diffeomorphisms with 2-dimensional expanding attractors It means that topological dimension of such attractor is equal to dimension of unstable manifold of any point belonging to attractor.

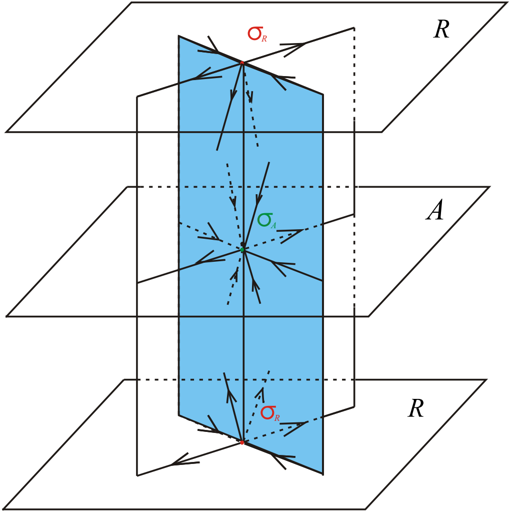


$n = 3$, M^3– closed orientable 3-manifold. Examples of basic sets $\mathcal{B}$:

Diffeomorphisms given on $\mathbb{T}^3$ with 2-dimensional attractor and repeller being 2-dimension tori. Restrictions of diffeomorphism to such basic set topologically conjugated with Anosov diffeomorphism.
It is clear that such basic set is not expanding attractor or attracting repeller.



$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



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σ_A
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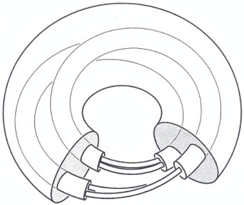
R

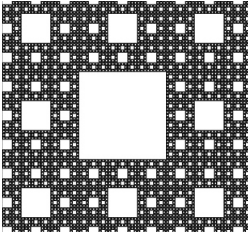
C. Bonatti problem. Topological structure of basic set of dimension 2

Theorem (Brown, 2010)

Any connected two-dimensional basic sets of diffeomorphisms of three-dimensional manifold is exactly one of the following:

1. *expanding attractor*
2. *attracting repeller*
3. *two-dimensional torus.*





Two-dimensional expanding attractors and topology of an ambient manifold M^3

Theorem (Grines, Zhuzhoma, 2004)

Let $f : M^3 \to M^3$ is structurally stable diffeomorphism, nonwandering set of which contains a two-dimensional expanding attractor. Then the manifold M^3 is diffeomorphic to the torus $\mathbb{T}^3$ and f is topologically conjugated with the diffeomorphism obtained from Anosov diffeomorphism by the generalized surgery operation.

Surface basic set

Definition

A basic set of diffeomorphism $f : M^3 \to M^3$ is called surface basic set if it belongs to a f-invariant closed 2-dimensional manifold M^2.

Theorem (Grines, Medvedev, Zhuzhoma 2005)

Let $f : M^3 \to M^3$ diffeomorphism, nonwandering set of which contains a connected two-dimensional surface attractor $\mathcal{B}$. Then $\mathcal{B} = M^2$, M^2 is tamly embedded torus, and the restriction f to M^2 is conjugated with an Anosov automorphism of the torus.

Remark

The two-dimensional torus may be no smooth at any point (Kaplan J, Mallet-Parret J, Yorke J, 1984).

Two-dimensional basic sets and the structure of the ambient manifold of M^3

Theorem (V.Z. Grines, V.S. Medvedev, Ya. A. Levchenko (2010))

Let $\Omega(f)$ of $f : M^3 \to M^3$ consists of a two-dimensional surface basic sets. Then M^3 is a locally trivial bundle over the circle with fiber homeomorphic to a two-dimensional torus.

The structure of the ambient manifold of M^3

Denote by M_τ quotient space obtained from $\mathbb{T}^2 \times [0,1]$ by identifying the points $(z,1)$ and $(\tau(z),0)$ where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ a homeomorphism.

The structure of the ambient manifold of M^3

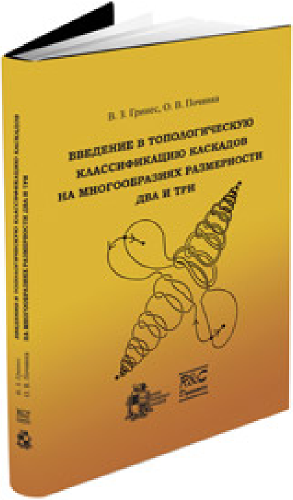
Theorem

If a closed oriented 3-manifold M^3
1) either is irreducible and admits diffeomorphism f such that there is f-invariant smooth torus T^2 and restriction f to T^2 induces hyperbolic automorphism in homology group [F. R. Hertz, M. A. R. Hertz, R. Ures (2011)]
2) or admits A-diffeomorphism f such that nonwandering set $\Omega(f)$ consists of 2-dimensional surface attractors and repellers [V. Grines, Yu. Levchenko, O. Pochinka (2013)]
Then M^3 is diffeomorphic to $M_{\widehat{J}}$, where $\widehat{J}$ algebraic automorphism of the torus given by the matrix J, which is either hyperbolic or coincides with the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or with the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Classification theorem

Theorem (V. Grines, Yu. Levchenko, V. Medvedev, O. Pochinka (2015))

Every A-diffeomorphism f such that the non-wandering set $\Omega(f)$ consists of 2-dimensional surface attractors and repellers is a locally direct product of a hyperbolic automorphism of the 2-torus and rough diffeomorphism of the circle.


ВВЕДЕНИЕ В ТОПОЛОГИЧЕСКУЮ
КЛАССИФИКАЦИЮ КАСКАДОВ
НА МНОГООБРАЗИЯХ РАЗМЕРНОСТИ
ДВА И ТРИ

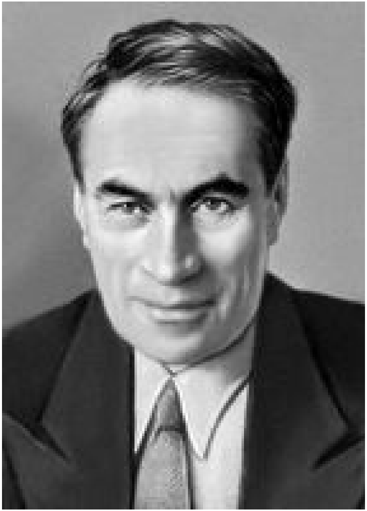


Figure: A. Andronov

back

Figure: L. Pontryagin

back

An equilibrium point x_0 where $v(x_0) = 0$, is said to be hyperbolic if none of the eigenvalues of the linearization of v at x_0 is purely imaginary. A periodic orbit of a flow is said to be hyperbolic if none of the eigenvalues of the Poincare return map at a point on the orbit has absolute value one.

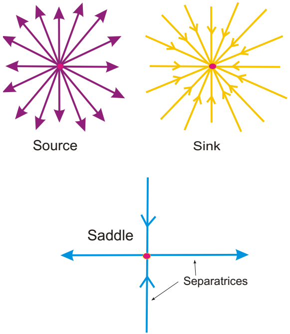


Figure: Hyperbolic equilibrium points

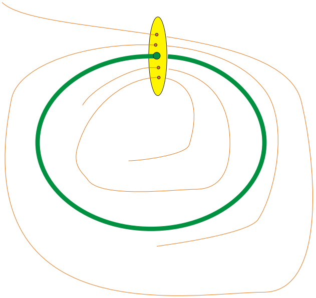


Figure: Hyperbolic periodic orbit

back

Saddle connection refers to a situation where an orbit from one saddle point enters the same (homoclinic orbit) or another saddle point (heteroclinic orbit), i.e. the unstable and stable saddle separatrices are connected.

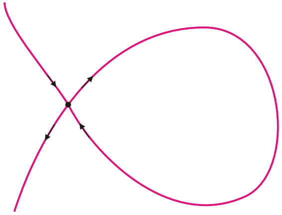


Figure: Homoclinic connection

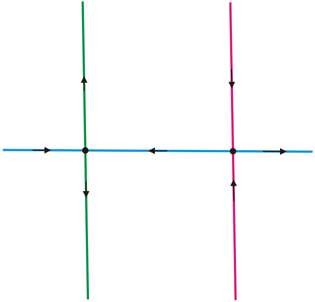


Figure: Heteroclinic connection

back

Figure: E. Leontovich-Andronova

back

Figure: A. Mayer

back

Figure: A. Mayer

back

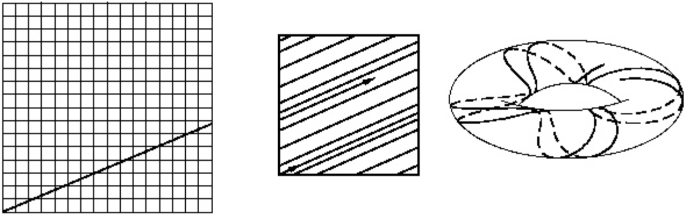


Figure: Irrational winding of the torus

back

Figure: M. Peixoto

back

Figure: M. Peixoto

back

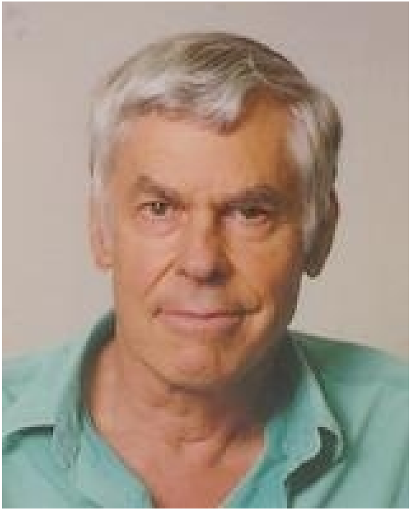


Figure: S. Smale

back

Figure: J. Palis

back

Figure: A. Oshemkov

back

Figure: V. Sharko

back

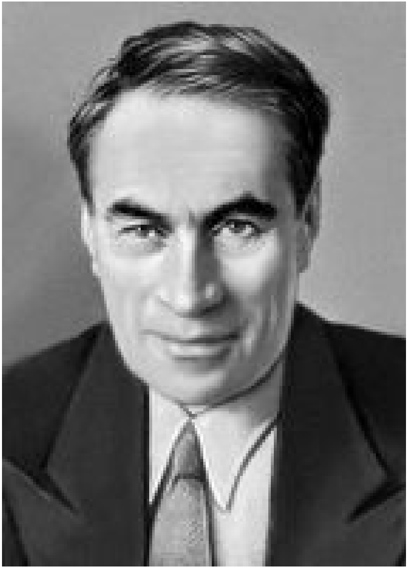


Figure: A. Andronov

back

Figure: E. Leontovich-Andronova

back

Figure: A. Mayer

back

Figure: M. Peixoto

back

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Figure: A. Bezdenezhnych

Figure: V. Grines

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Figure: G. Fleitas

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Figure: Ya. Umanskii

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Figure: S. Pilyugin

back

Figure: V. Grines

back

Figure: E. Gurevich

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Figure: D. Pixton

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Figure: Cr. Bonatti

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Figure: R. Plykin

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Figure: V. Grines

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Figure: Cr. Bonatti

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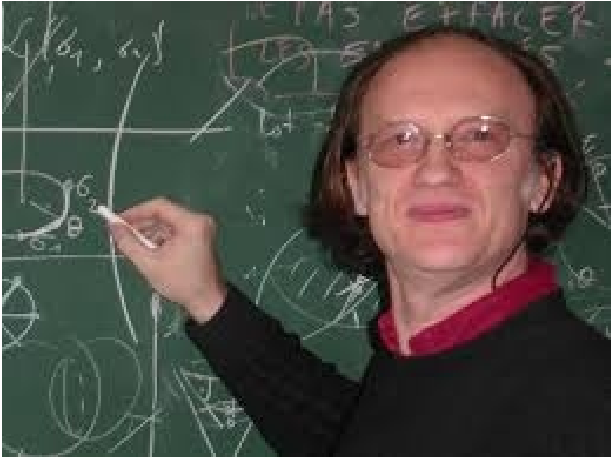

Figure: R. Langevin

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Figure: R. Thom

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Figure: Ya. Sinai

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Figure: D. Anosov

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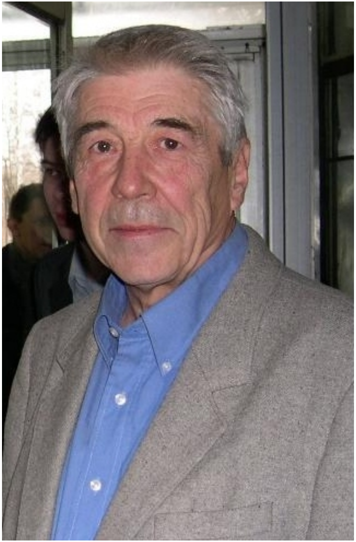

Figure: L. Shilnikov

back