Lagrangian Manifolds and Complex Vector Bundles, Corresponding to Asymptotic Solutions for Equations with Delta-type Singularities

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Outline

Cauchy problem for ε-pseudodifferential evolutionary equation

$$i\varepsilon\frac{\partial u}{\partial t} = H(x, -i\varepsilon\frac{\partial}{\partial x})u, \quad x \in \mathbb{R}^n, \varepsilon \to +0,$$

$H(x,p) : \mathbb{R}^{2n} \to \mathbb{R}$ is smooth.

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{\varepsilon}}, \quad S_0 \in C^\infty, \varphi^0 \in C_0^\infty.$$

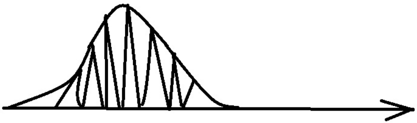

Рис.: Wave packet

Rapidly oscillating wave packet - S_0 is real. Asymptotic solution. Consider initial Lagrangian surface $\Lambda_0 \subset \mathbb{R}^{2n}$, $p = \frac{\partial S_0}{\partial x}$ and shift it by the flow g_t of the classical Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \Lambda_t = g_t \Lambda_0.$$

Volume form $\sigma_0 = dx$ on Λ_0, $\sigma_t = g_t^* dx$ on Λ_t

Theorem

(V.P. Maslov, $\sim$ 1965). Under certain technical conditions the solution $u(x,t,h)$ can be represented as asymptotic serie

$$u \sim K_{\Lambda_t,\sigma_t}\Big(\sum_{k=0} \varepsilon^k \varphi_k\Big),$$

$K : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}^n_x)$ is the Maslov canonical operator, φ_k are smooth functions on Λ_t, $\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

Рис.: Squeezed state

Localized ("squeezed") initial state $S_0(x)$ is complex, $\Im S_0 \geq 0$, $\Im S_0 = 0$ on the smooth k-dimensional surface W_0, $d^2 \Im S_0|_{NL_0} > 0$. Consider k-dimensional isotropic surface $\Lambda_0 \subset \mathbb{R}^{2n}$: $x \in W_0, p = \frac{\partial S_0}{\partial x}$ and n-dimensional complex vector bundle ρ_0 over Λ_0 (Maslov complex germ): fiber $\rho(x,p)$ is the plane in ${}^{\mathbb{C}}T_{x,p}\mathbb{R}^{2n}$, $\xi_p = \frac{\partial^2 S_0}{\partial x^2}\xi_x$. Shifted bundle $\Lambda_t = g_t \Lambda_0$, $\rho_t = dg_t \rho_0$.

Theorem

Under certain technical conditions the solution $u(x, t, h)$ can be represented as asymptotic serie

$$u \sim \hat{K}_{\Lambda_t, \rho_t}(\sum_{k=0} \varepsilon^k \varphi_k),$$

$\hat{K} : C_0^\infty(\Lambda_t) \to C^\infty(\mathbb{R}_x^n)$ is the Maslov canonical operator on the complex germ, φ_k are smooth functions on Λ_t,
$\varphi_0(\alpha) = \varphi^0(g_{-t}\alpha)$.

Simplest case:

$$S_0 = (p_0, x - x_0) + \frac{1}{2}(x - x_0, Q_0(x - x_0))), \quad p_0 \in \mathbb{R}^n, Q^t = Q, \Im Q > 0.$$

W_0 is the point x_0, $\rho_0 : \xi_p = Q_0 \xi_x$.

$$u(x, t, h) \sim e^{\frac{iS(x,t)}{\varepsilon}} \sum_{k=0}^{\infty} (\varepsilon^k \varphi_k(x, t)).$$

$$S = q(t) + (P(t), x - X(t)) + \frac{1}{2}(x - X(t), Q(t)(x - X(t))),$$

$$\dot{X} = \frac{\partial H}{\partial p}, \quad \dot{P} = -\frac{\partial H}{\partial x}.$$

Problem

What happens if coefficients of initial equation contain singularities?

$$\mathrm{i}\varepsilon\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = -\frac{\varepsilon^2}{2}\Delta \mathrm{u} + \mathrm{V}(\mathrm{x})\mathrm{u} + \mathrm{q}(\mathrm{x})\delta_{\mathrm{M}}\mathrm{u},$$

$$\mathrm{u}|_{\mathrm{t}=0} = \varphi^0 \mathrm{e}^{\frac{\mathrm{i}\mathrm{S}_0}{\varepsilon}}$$

M is a smooth oriented hypersurface, S_0 is real. Boundary conditions on M:

$$\mathrm{u}_-|_{\mathrm{M}} = \mathrm{u}_+|_{\mathrm{M}}, \varepsilon \quad \frac{\partial \mathrm{u}}{\partial \mathrm{m}_-}|_{\mathrm{M}} - \varepsilon\frac{\partial \mathrm{u}}{\partial \mathrm{m}_+}|_{\mathrm{M}} = \mathrm{q}\mathrm{u}|_{\mathrm{M}}$$

Extended phase space $\mathbb{R}^{2n+2}_{(x,t,p,p_0)}$. Isotropic surface Λ_0: $t = 0, p = \frac{\partial S_0}{\partial x}, H = 0$, $H = p_0 - \frac{1}{2}|p|^2 - V(x)$, Lagrangian manifold $\Lambda^+ = \bigcup_s g_s\Lambda_0$.
Hypersurface $\hat{M} \subset \mathbb{R}^{2n+2}$, $x \in M$. $N^+ = \Lambda \bigcap \hat{M}$. For $x \in M$ let p_τ denote the projection of p to T_xM, p_n – normal component.
Map $Q : \hat{M} \to \hat{M}$, $Q(x, t, p_\tau, p_n, p_0) = (x, t, p_\tau, -p_n, p_0)$, $N^- = Q(N^+)$. Reflected Lagrangian manifold $\Lambda^- = \bigcup_s g_sN^-$.

Volume form. On Λ_0 we have $\sigma_0 = \mathrm{dx}$, construct invariant form on Λ^+: $\sigma^+(\alpha, \mathrm{s}) = \mathrm{g}_\mathrm{s}^* \sigma_0 \wedge \mathrm{ds}$. On N^+ consider $\mathrm{i}_{\mathrm{p_n}} \sigma^+$, map it to N^- and construct invariant form σ^-.

Consider formal series

$$u = K_{\Lambda^+}(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k^+) + K_{\Lambda^-}(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k^-)$$

on the negative side of M,

$$u = K_{\Lambda^-}(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k^*)$$

on the positive side.

$$\varphi_0^*|_{N^+} = \frac{2ip_n}{2ip_n + q} \varphi_0^+|_{N^+}, \quad \varphi_0^-|_{N^-} = \frac{-q}{q + 2ip_n} \varphi_0^+|_{N^+}$$

Theorem

This series is asymptotic for the solution of the Cauchy problem for $t \in [0, T]$.

Remark

$$\tau = \frac{2ip_n}{2ip_n + q}, \quad r = \frac{-q}{q + 2ip_n}$$

are the analogs of the coefficients of transmission and reflection.

Reflection of vector bundles
Rules of reflection

The fibers are positive complex Lagrangian planes – quadratic forms on $T_P\mathbb{R}^n$. On $T_P M$ it is shifted by $p_n b$, where b is the second fundamental form of M, on the pair (m, ξ) — be the value $p_n \partial_\xi(V)$, on the pair (m, m) – by $p_n^2 \partial_m(V)$.

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\Phi(x)}{\varepsilon}, x\right)\Delta u, \quad x \in \mathbb{R}^n,$$

$\varepsilon \to 0$.
$\Phi(x) : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, and the equation $\Phi = 0$ defines a smooth regular hypersurface $M \subset \mathbb{R}^n$
$c(y, x) : \mathbb{R}^{n+1} \to \mathbb{R}$ is smooth and strictly positive,
$c(y, x) \to c^+(x)$ as $y \to \infty$ and $c(y, x) \to c^-(x)$ as $y \to -\infty$.
Initial conditions

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{\varepsilon}}, \quad u_t|_{t=0} = 0.$$

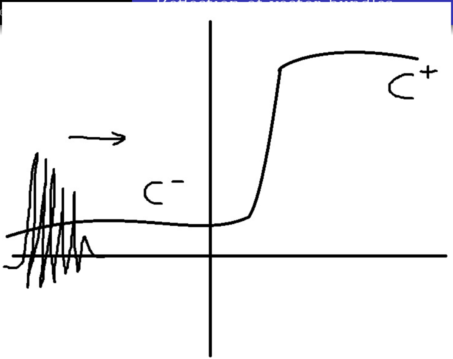

Рис.: Scattering

Fast variable $y = \Phi(x)/\varepsilon$

$$c^2\Delta \to \frac{1}{\varepsilon^2}c^2(y,x)\left(\varepsilon\nabla + \nabla\Phi\frac{\partial}{\partial y}\right)^2,$$

Leading ε-symbol

$$-c^2(x,y)(p - i\nabla\Phi\frac{\partial}{\partial y})^2.$$

Standard scheme for the case of discrete spectrum classical Hamiltonians are eigenvalues $\lambda(p,x)$ of the symbol.
In our case, the spectrum of the symbol contains a continuous component.

One-dimensional case.
Let $x \in \mathbb{R}$, M is a point x_0; $\Phi(x) = x - x_0$.

Regular velocity $c = c^-(x)$.

Assertion

For $n = 1$ and $c = c^-(x)$, the solution of the Cauchy problem can be expanded in the asymptotic series

$$u \sim \sum_{1,2} e^{\frac{iS^{1,2}(x,t)}{\varepsilon}} \sum_{k=0}^{\infty} \varepsilon^k \varphi_k^{1,2}(x,t),$$

$$S^{2,1} = S_0(z(x,\pm t)), \quad \varphi_0^{2,1} = \frac{1}{2}\sqrt{\frac{c(x)}{c(z(x,\pm t))}}\varphi^0(z(x,\pm t)),$$

where $z(x,t)$ is found from the equation

$$t = \int_z^x \frac{d\xi}{c_0(\xi)}.$$

Remark

The summands with thee superscript "2" describe the wave propagating to the right and those with the superscript "1" to the left.

Remark

The corrections φ_{k} are expressed by explicit formulas using $\mathrm{S}, \varphi_1, \ldots, \varphi_{\mathrm{k}-1}$.

Remark

If c contains a rapidly changing part ($c = c(\frac{x-x_0}{\varepsilon}, x)$), then the solution of the Cauchy problem still has the indicated form for sufficiently small times, until the initial packet had time to reach the zone of rapid speed changes. Times like this are given by the inequality

$$t \le \min_{z \in \operatorname{supp} \varphi^0} \int_z^{x_0} \frac{dx}{c(x)} - \delta,$$

where δ is an arbitrarily small positive number independent of ε.

Rapidly varying velocity. When the wave packet reaches the point x_0, it splits into two packets, the transmitted and reflected ones. The reflection process is described by the second-order ordinary differential equation

$$-\frac{d^2 g}{dy^2} + V(y,t)g = 0,$$

where $V = \frac{k^2(c^-)^2}{c^2}|_{x=x_0} = \frac{\chi^2(c^+)^2}{c^2}|_{x=x_0}$; here $k = \frac{\partial S^-}{\partial x}$ and $\chi = \frac{\partial S^+}{\partial x}$, and $S^\pm$ are the phases of the transmitted and reflected waves; explicit formulas for them are given below. Note that $V \to V^\pm$ as $y \to \pm\infty$, where $V^+ = \chi^2$, $V^- = k^2$; this equation has a unique solution $g_0(y,t)$ with the following behavior at infinity:

$$g_0 \to e^{iky} + re^{-iky} \quad \text{as} \quad y \to -\infty; \quad g_0 \to \tau e^{i\chi y} \quad \text{as} \quad y \to +\infty.$$

Here $r(t)$ and $\tau(t)$ are called the (complex) reflection and transmission coefficients, respectively.

Theorem

On every finite closed interval, the solution of the Cauchy problem can be expanded in the asymptotic series

$$u \sim \sum_{k=0}^{\infty} \varepsilon^k [e^{\frac{iS^1(x,t)}{\varepsilon}} \varphi_k^1(x,t) + e^{\frac{iS^2(x,t)}{\varepsilon}} f_k^2\left(\frac{x-x_0}{\varepsilon}, x, t\right) +$$

$$+ e^{\frac{iS^+(x,t)}{\varepsilon}} f_k^+\left(\frac{x-x_0}{\varepsilon}, x, t\right) + e^{\frac{iS^-(x,t)}{\varepsilon}} f_k^-\left(\frac{x-x_0}{\varepsilon}, x, t\right)],$$

where

$$S^{\pm}(x,t) = S_0(w^{\pm}(x,t)),$$

$w^{\pm}$satisfies the equation

$$\int_{w^{\pm}}^{x_0} \frac{dx}{c^-(x)} \pm \int_{x_0}^{x} \frac{dx}{c^{\pm}(x)} = t;$$

$$S^2 = S_0(z(x_0, t - \int_{x_0}^{x} \frac{dx}{c^-(x)}));$$

$$f_k^{\pm}(y, x) \to \varphi_k^{\pm}(x, t) \quad \text{as} \quad y \to \pm\infty, \quad f_k^{\pm}(y, x) \to 0 \quad \text{as} \quad y \to \mp\infty,$$

$$f_k^{2}(y, x) \to \varphi_k^{2}(x, t) \quad \text{as} \quad y \to -\infty, \quad f_k^{2}(y, x) \to 0 \quad \text{as} \quad y \to +\infty.$$

The functions $\varphi_k^{\pm,1,2}$ are smooth; they are evaluated explicitly.

The leading part of the asymptotics:

$$\varphi_0^2(x,t) = \frac{1}{2}\varphi_0(z(x,t))\sqrt{\frac{c^-(x)}{c^-(z(x,t))}},$$

$$\varphi_0^+(x,t) = \tau\left(t - \int_{x_0}^{x}\frac{dx}{c^+(x)}\right)\varphi_0^2\left(x_0, t - \int_{x_0}^{x}\frac{dx}{c^+(x)}\right)\sqrt{\frac{c^+(x)}{c^+(x_0)}},$$

$$\varphi_0^-(x,t) = r\left(t + \int_{x_0}^{x}\frac{dx}{c^-(x)}\right)\varphi_0^2\left(x_0, t + \int_{x_0}^{x}\frac{dx}{c^-(x)}\right)\sqrt{\frac{c^-(x)}{c^-(x_0)}},$$

$$f_0^+(y, x_0, t) = \varphi_0^+(x, t)\eta(y), \quad f_0^2(y, x_0, t) = \varphi_0^2(x, t)(1-\eta(y)) + \hat{g}(y, t),$$

$$f_0^-(y, x_0, t) = \varphi_0^-(1-\eta(y)), \eta(y) = \frac{1}{2}(1+\tanh y),$$

$$\hat{g}(y, t) = \varphi_0^2(x_0, t)[g_0(y, t)e^{-iky} - (1-\eta) - \tau(t)\eta e^{iy(\chi-k)} - r(t)(1-\eta)e^{-2iky}].$$

Here $z(x, t)$ is the solution of the equation

$$\int_z^x \frac{d\xi}{c^-(\xi)} = t.$$

Remark

The asymptotics of the solution consists of four wave packets: the first, marked with “1”, propagates to the left and does not interact with the rapidly changing part of the speed. The wave propagating in the direction of the point x_0, marked with “2”, is divided after reaching this point into two waves: the transmitted (marked with “+”) and reflected (marked with ‘ -”). All waves move with the unperturbed speed; the waves to the left of the point x_0 move with the speed c^-; the wave to the right of x_0 (i.e., transmitted) moves with with the speed c^+; at the point x_0, the amplitudes of the transmitted and reflected waves are proportional to the coefficients τ and r, respectively.

The phases S satisfy the Hamilton–Jacobi equations

$$\frac{\partial S^{+,2}}{\partial t} + c^{+}(x)\frac{\partial S^{+,2}}{\partial x} = 0, \quad \frac{\partial S^{-,1}}{\partial t} - c^{-}(x)\frac{\partial S^{-,1}}{\partial x} = 0;$$

here, for $S^{1,2}$, the initial conditions are posed at $t = 0$, and for $S^{\pm}$ a boundary condition is posed at x_0: $S^{1,2}|_{t=0} = S_0(x)$, $S^{\pm}|_{x=x_0} = S^2|_{x=x_0}$.
At the point x_0, $c^{-}\frac{\partial S^2}{\partial x} = -c^{-}\frac{\partial S^{-}}{\partial x} = c^{+}\frac{\partial S^{+}}{\partial x}$ (the reflection).

The amplitudes $\varphi_0^{\pm,1,2}$ satisfy the transport equations

$$\frac{\partial S^{-,1,2}}{\partial t}\frac{\partial \varphi_0^{-,1,2}}{\partial t} - c^{-2}(x)\frac{\partial S^{-,1,2}}{\partial x}\frac{\partial \varphi_0^{-,1,2}}{\partial x} +$$

$$+\frac{1}{2}\left(\frac{\partial^2 S^{-,1,2}}{\partial t^2} - c^{-2}(x)\frac{\partial^2 S^{\pm,1,2}}{\partial x^2}\right)\varphi_0^{\pm,1,2} = 0,$$

$$\frac{\partial S^{+}}{\partial t}\frac{\partial \varphi_0^{+}}{\partial t} - c^{+2}(x)\frac{\partial S^{+}}{\partial x}\frac{\partial \varphi_0^{+}}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 S^{+}}{\partial t^2} - c^{+2}(x)\frac{\partial^2 S^{+}}{\partial x^2}\right)\varphi_0^{+} = 0.$$

For the functions $\varphi_0^{1,2}$, the conditions are posed at $\mathrm{t} = 0$ and, for the functions $\varphi_0^{\pm}$, at the point x_0:

$$\varphi_0^{+}|_{\mathrm{t}=0} = \varphi_0^{-}|_{\mathrm{t}=0} = \frac{1}{2}\varphi^0(\mathrm{x}),$$

$$\varphi_0^{-}|_{\mathrm{x}=\mathrm{x}_0} = \mathrm{r}(\mathrm{t})\varphi_0^2|_{\mathrm{x}=\mathrm{x}_0}, \quad \varphi_0^{+}|_{\mathrm{x}=\mathrm{x}_0} = \tau(\mathrm{t})\varphi_0^2|_{\mathrm{x}=\mathrm{x}_0}.$$

Multidimensional case. Two new effects.

1. Focal points. Use of Maslov canonical operator.

2. Total reflection. Transmitted wave can dissapear.

Рис.: Total reflection

Regular velocity $c = c^-$. In $\mathbb{R}^{2n+2}$ with the coordinates (x, t, p, p_0), $p \in \mathbb{R}^n$, $p_0 \in \mathbb{R}$, consider a smooth n-dimensional isotropic surface Λ_0 given by the equations $x \in U$, $p = \nabla S_0$, $t = 0$, $p_0^2 - c_0^2|p|^2 = 0$, where U is an arbitrary neighborhood of the support of the function φ^0; this surface consists of two components $\Lambda_0^{1,2}$ given by the conditions $(p_0 \mp c_0|p|)|_{\Lambda_0^{1,2}} = 0$. Let us move the surface Λ_0 along the trajectories of the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{t} = \frac{\partial H}{\partial p_0}, \quad \dot{p}_0 = -\frac{\partial H}{\partial t} = 0,$$

where $H(x, t, p, p_0) = p_0^2 - (c^-)^2(x)|p|^2$; let $\Lambda = \bigcup_{s\in\mathbb{R}} g_s\Lambda_0$. We assume that T is such that the intersection of Λ with the domain $x \in U$, $t \in (-\delta, T + \delta)$ for some $\delta > 0$ is a smooth submanifold; it also consists of two components $\Lambda^{1,2}$ (obtained by shifting $\Lambda_0^{1,2}$).

Assertion

For $t \in [0, T]$, the solution of the Cauchy problem is expanded in the asymptotic series

$$u = K_{\Lambda}\Big(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\Big),$$

where K stands for the canonical Maslov operator, φ_k are smooth functions on Λ; the function φ_0 is obtained from the initial function φ^0 by a shift along the trajectories of the Hamiltonian system:

$$\varphi_0(\alpha) = \frac{1}{2}\pi^* \varphi^0(g_{-s}\alpha), \quad \alpha \in \Lambda, g_{-s}\alpha \in \Lambda_0, \pi$$

is the natural projection $\mathbb{R}^{2n+2}_{(x,t,p,p_0)} \to \mathbb{R}^n_x$.

Let $c = c(\frac{\Phi(x)}{\varepsilon}, x)$; we assume that $\nabla\Phi \neq 0$ in $\mathbb{R}^n$ and, moreover, $|\nabla\Phi| = 1$ in a neighborhood of M (i.e., Φ is the distance to M along the normal). Consider two Hamiltonian systems

$$\dot{x} = \frac{\partial H^{\pm}}{\partial p}, \quad \dot{p} = -\frac{\partial H^{\pm}}{\partial x}, \quad \dot{t} = \frac{\partial H^{\pm}}{\partial p_0}, \quad \dot{p}_0 = -\frac{\partial H^{\pm}}{\partial t} = 0, \qquad (1)$$

where $H^{\pm}(x, t, p, p_0) = p_0^2 - (c^{\pm})^2(x)|p|^2$;
Suppose that the projections of the trajectories of these systems to $\mathbb{R}^n_x$ are transversal to M; consider in $\mathbb{R}^{2n+2}$ the surface $\hat{M}$ given by the equation $\Phi(x) = 0$ (the lifting of M to the phase space), and let $N^2 = \Lambda^2 \bigcap \hat{M}$; we assume that N^2 is a smooth connected surface. For every $x \in M$, denote by p_τ the projection of the vector p to the tangent plane T_xM, and by p_n the normal component of this vector ($p_n = (p, \nabla\Phi)$).

We assume that on the surface N^2, for some $\delta > 0$, one of the two conditions is satisfied

1. Refraction
$$(\mathrm{c}^-)^2|\mathrm{p}|^2 - (\mathrm{c}^+)^2\mathrm{p}_\tau^2 \geq \delta,$$
or
2. Total reflection
$$(\mathrm{c}^-)^2|\mathrm{p}|^2 - (\mathrm{c}^+)^2\mathrm{p}_\tau^2 \leq -\delta < 0.$$

First case.
Consider the mappings $Q^{\pm} : \hat{M} \to \hat{M}$ defined by the formulas $Q^{\pm}(x, t, p_\tau, p_0) = (x, t, p_\tau, p_n^{\pm})$, $p_n^- = -p_n$, $p_n^+ = (\frac{(c^-)^2|p|^2-(c^+)^2p_\tau^2}{(c^+)^2})^{1/2}$, and let $N^{\pm} = Q^{\pm}(N^2)$. We move $N^{\pm}$ along the trajectories of the Hamiltonian systems; we obtain surfaces $\Lambda^{\pm} = \bigcup_{s\in\mathbb{R}} g_s^{\pm} N^{\pm}$. We assume that the intersections of $\Lambda^{\pm,2}$ with the domain $t \in (-\delta, T + \delta)$ are smooth $(n + 1)$-dimensional submanifolds, and $\Lambda^{\pm,2} \bigcap \hat{M} = N^{\pm,2}$. Moreover, suppose that in the domain $t \in (-\delta, T + \delta)$ we have $\Lambda^1 \bigcap \hat{M} = \emptyset$ (the corresponding trajectories go away from M). The surfaces $\Lambda^{1,2}, \Lambda^{\pm}$ define the asymptotics of the solution to the Cauchy problem.

The construction of the canonical operator involves the volume form on the Lagrangian manifold; we construct such forms on $\Lambda^{\pm,1,2}$. On the starting surface Λ_0 we define the volume form $d\sigma_0 = dx_1 \wedge \dots dx_n$ and extend it to volume forms $d\sigma^{1,2}$ on $\Lambda^{1,2}$ invariant with respect to the trajectories (if $\alpha \in \Lambda$, $\alpha = g_s^- \alpha_0$, $\alpha_0 \in \Lambda_0$, then $d\sigma(\alpha) = (g_s^-)^* d\sigma_0 \wedge ds$). On the surface N^2 we consider the form $d\sigma/d\Phi$, transfer it to $N^\pm$ by the diffeomorphisms $Q^\pm$, and extend these forms again to invariant volume forms $d\sigma^\pm$ on $\Lambda^\pm$. Thus, invariant volume forms are defined on four invariant Lagrangian manifolds $\Lambda^{\pm,1,2}$.

In the second case, we will consider only the surface N^{-} with the form $\mathrm{d}\sigma^{-}$ (the mapping Q^{+} in this case is not defined)

Consider a second-order differential equation with respect to the variable y whose coefficients depend on the point α of the surface N^2,

$$-\frac{\mathrm{d}^2\mathrm{g}}{\mathrm{dy}^2}+\mathrm{V}(\alpha)\mathrm{g}=0,\quad \mathrm{V}=|\mathrm{p_n}|^2+|\mathrm{p}|^2(\frac{(\mathrm{c}^-)^2(\mathrm{x})}{\mathrm{c}^2(\mathrm{y},\mathrm{x})}-1)|_{\mathrm{M}}.$$

In the case 1 there exists a solution g_0 such that

$$g_0 \to e^{ip_n y} + re^{-ip_n y} \quad \text{as} \quad y \to -\infty; \quad g_0 \to \tau e^{ip_n^+ y} \quad \text{as} \quad y \to +\infty,$$

$$p_n^+ = (\frac{(c^-)^2|p|^2 - (c^+)^2 p_\tau^2}{(c^+)^2})^{1/2},$$

while in the case 2

$$g_0 \to e^{ip_n y} + re^{-ip_n y} \quad \text{as} \quad y \to -\infty; \quad g_0 \to \tau e^{-\varkappa y} \quad \text{as} \quad y \to +\infty,$$

$$\varkappa = (\frac{(c^+)^2 p_\tau^2 - (c^-)^2|p|^2}{(c^+)^2})^{1/2},$$

$\tau(\alpha)$ и $r(\alpha)$ — coefficients of transmission and reflection.

Theorem

Case 1. For $t \in [0, T]$, the solution to the Cauchy problem is expanded in the asymptotic series

$$u = K_{\Lambda^1}\left(\sum_{k=0}^{\infty} \varepsilon^k f_k^1(\alpha)\right) + K_{\Lambda^+}\left(\sum_{k=0}^{\infty} \varepsilon^k f_k^+\left(\frac{\Phi(x)}{\varepsilon}, \alpha\right)\right) +$$

$$+K_{\Lambda^-}\left(\sum_{k=0}^{\infty} \varepsilon^k f_k^-\left(\frac{\Phi(x)}{\varepsilon}, \alpha\right)\right) + K_{\Lambda^2}\left(\sum_{k=0}^{\infty} \varepsilon^k f_k^2\left(\frac{\Phi(x)}{\varepsilon}, \alpha\right)\right),$$

α is the point of the corresponding manifold, K stands for the Maslov canonical operator, the functions f_k^1 are the same as in the previous assertion,

$$f_k^{\pm}(y,\alpha) \to \varphi_k^{\pm}(\alpha,t) \quad \text{as} \quad y \to \pm\infty, \quad f_k^{\pm}(y,\alpha) \to 0 \quad \text{as} \quad y \to \mp\infty,$$
$$f_k^2(y,\alpha) \to \varphi_k^2(\alpha,t) \quad \text{as} \quad y \to -\infty, \quad f_k^2(y,\alpha) \to 0 \quad \text{as} \quad y \to +\infty,$$

the functions φ_k^2 are the same as in the previous assertion. The leading part of the asymptotic is as follows:

$$\varphi_0^{+}(\alpha) = \varphi_0^2(\alpha)\tau(\alpha'), \quad \varphi_0^{-}(\alpha) = \varphi_0^2(\alpha)r(\alpha'),$$
$$f_0^{+}(y,\alpha) = \varphi_0^{+}(\alpha)\eta(y), \quad f_0^{-}(y,\alpha) = \varphi_0^{-}(\alpha)(1-\eta(y)) + \hat{g}(y,\alpha'),$$

where

$$\eta(y) = \frac{1}{2}(1+\tanh y), \quad \hat{g}(y,\alpha') = \varphi_0^2(\alpha')[g_0(y,\alpha')e^{-iky} -$$
$$-(1-\eta) - \tau(\alpha')\eta e^{iy(\chi-k)} - r(\alpha')(1-\eta)e^{-2iky}].$$

Here $\alpha' \in N^{\pm}, \alpha \in \Lambda^{\pm}$ and $\alpha = g_s\alpha'$.

Theorem

Case 2.

$$u = K_{\Lambda^1}\left(\sum_{k=0}^{\infty}\varepsilon^k f_k^1(\alpha)\right) + K_{\Lambda^-}\left(\sum_{k=0}^{\infty}\varepsilon^k f_k^-\left(\frac{\Phi(x)}{\varepsilon},\alpha\right)\right) +$$

$$+K_{\Lambda^2}\left(\sum_{k=0}^{\infty}\varepsilon^k f_k^2\left(\frac{\Phi(x)}{\varepsilon},\alpha\right)\right),$$

$$f_k^{-,2}(y,\alpha) \to \varphi_k^{-,2}(\alpha,t) \quad \text{as} \quad y \to -\infty,$$

$$f_k^{-,2}(y,\alpha) \to 0 \quad \text{as} \quad y \to +\infty,$$

$$\varphi_0^-(\alpha) = \varphi_0^2(\alpha)\mathrm{r}(\alpha'), \quad \mathrm{f}_0^- = \varphi_0^-(1-\eta(\mathrm{y})),$$

$$\mathrm{f}_0^2(\mathrm{y},\alpha) = \varphi_0^-(\alpha)(1-\eta(\mathrm{y})) + \hat{\mathrm{g}}(\mathrm{y},\alpha'),$$

$$\eta(\mathrm{y}) = \frac{1}{2}(1+\tanh\mathrm{y}), \quad \hat{\mathrm{g}}(\mathrm{y},\alpha') = \varphi_0^2(\alpha')[\mathrm{g}_0(\mathrm{y},\alpha')\mathrm{e}^{-\mathrm{i}\mathrm{p}_\mathrm{n}\mathrm{y}} -$$

$$-(1-\eta) - \mathrm{r}(\alpha')(1-\eta)\mathrm{e}^{-2\mathrm{iky}}],$$

$\alpha' \in \mathrm{N}^-, \alpha \in \Lambda^-, \alpha = \mathrm{g}_\mathrm{s}\alpha'.$

THANK YOU

FOR YOUR

ATTENTION!