О графах с высокой регулярностью

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Декабрьские чтения в Томске

Томск, Россия

6-11 декабря 2021

Outline of talk

The main goals:

♢ to present generazilations of strongly regular graphs

♢ to show relationships with geometrical/topological aspects

Content

◇ Strongly regular graphs: historical background (1963)

◇ Deza graphs: a generalization of strongly regular graphs (1994, 1999)

◇ Strongly Deza graphs: a new concept (2021)

◇ Neimaier graphs: another generalization of strongly regular graphs

Part I

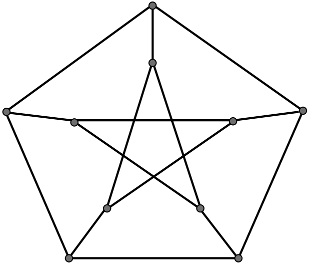
Strongly regular graphs

Strongly regular graphs

Definition

Let $G = (V, E)$ be a regular graph with n vertices and degree k. Then G is a strongly regular graph with parameters $SRG(n, k, \lambda, \mu)$ if:

- every two adjacent vertices have λ common neighbours
- every two non-adjacent vertices have μ common neighbours.



Petersen graph is $SRG(10, 3, 0, 1)$

SRGs and partial geometries: 1963

Original paper by Raj Chandra Bose, 1963

R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* 13 (1963) 389–419.

Partial geometry

A partial geometry $pg(s,t,\alpha)$ with parameters s,t,α, $\alpha \geqslant 1$ is an incidence structure of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines, two distinct lines meet in at most one point and given a line and a point not in it, there are exactly α lines through the point which meet the line.

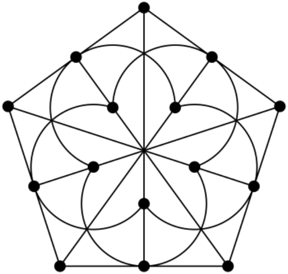
The point graph of $pg(s,t,\alpha)$ as SRG

- the points of the geometry are the vertices of G;
- two vertices of G are adjacent if and only if the corresponding points are incident with the same line of the geometry.

Parameters of SRG through a partial geometry

- $s(t+1)$-regular graph;
- if $\alpha = s+1$ then $G \cong K_n$
- if $\alpha \leqslant s$ then G is SRG (pseudo geometric graph) with parameters $\{n, s(t+1), s-1+t(\alpha-1), \alpha(t-1)\}$ where $n = (s+1)(st+\alpha)/\alpha$.

The smallest such a graph has parameters $(15, 6, 1, 3)$ and corresponds to:



partial geometry $pg(2,2,1)$

Partial geometries, SRG, and behind

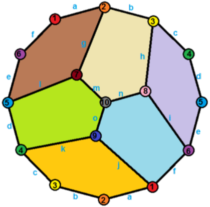
• $pg(2,2,1)$ is an example of a near polygon that is an incidence geometry introduced by Ernest Shult and Arthur Yanushka in 1980;

• there is a connection between near polygons and dual polar spaces;

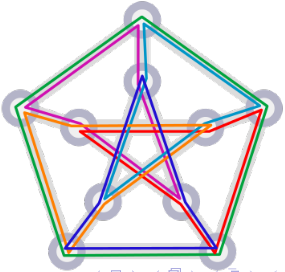
• some sporadic simple groups (the Mathieu groups) act as automorphism groups of near polygons; in particular, the M_{24} near hexagon related to the Mathieu group M_{24} and the extended binary Golay code. It is constructed by taking the 759 octads in the Witt design $S(5,8,24)$ corresponding to the Golay code as points and a triple of three pairwise disjoint octads as lines.

• the Petersen graph can be embedded without crossings is the projective plane; this construction forms a regular map and shows that the Petersen graph has non-orientable genus 1.

The Petersen graph and hemi-dodecahedron



Hemi-dodecahedron. The six faces of the hemi-dodecahedron depicted as colored cycles in the Petersen graph:



Strongly regular graphs: applications

- ◇ partial geometries (Bose, 1963; van Lint+, 1981; Švob+, 2020)
- ◇ rank 3 permutation groups (Higman, 1964)
- ◇ regular 2-graphs (Higman, Taylor, 1977)
- ◇ local graphs (Hall, 1985)

Main new book

Andries E. Brouwer, Hendrik Van Maldeghem, *Strongly regular graphs*.

https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf

Classical old book

A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin (1989) (Chapter 1).

Strongly regular graphs: properties

◇ *SRG* is a graph of diameter 2

◇ if $\mu \neq 0$ then the parameters of *SRG* are in the following relationship:

$$n = 1 + k + \frac{k(k - \lambda - 1)}{\mu}$$

◇ *SRG* with an adjacency matrix A is represented by matrix equations:

$$AJ = JA = kJ, \qquad A^2 = kI + \lambda A + \mu(J - I - A)$$

Spectral characterization of SRG

Any *SRG* has precisely three eigenvalues. Any connected regular graph with only three distinct eigenvalues is *SRG*.

Deza graphs

Deza graphs: generalization of *SRG*

Definition

A Deza graph G with parameters (n, k, b, a) is a k-regular graph of order n for which the number of common neighbours of two vertices takes values b or a, where $b \geqslant a$, whenever G is not the complete or the edgeless graph.

Original paper by Deza & Deza, 1994

A. Deza, M. Deza, The ridge graph of the metric polytope and some relatives. In: *Polytopes: Abstract, Convex and Computational*. NATO ASI Series, Vol. 440 (1994) 359–372, Springer.

Founding Deza graph theory, 1999

M. Erickson, S. Fernando, W.H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, *J. Combinatorial Design*, 7 (1999) 359–405.

Deza graphs: matrix representation and children

Let G be a graph with n vertices, and M be its adjacency matrix. Then G is a Deza graph with parameters (n, k, b, a) if and only if

$$M^2 = aA + bB + kI$$

for some symmetric $(0, 1)$-matrices A and B such that

$$A + B + I = J,$$

where J is the all-ones matrix and I is the identity matrix.

Definition 1

Let G be a Deza graph with M, A and B such that

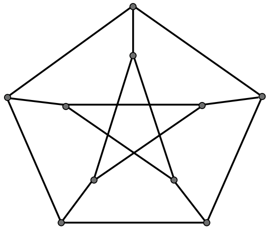
$$M^2 = aA + bB + kI.$$

Then A and B are adjacency matrices of graphs, and the corresponding graphs G_A and G_B are the *children* of G.

Deza graphs: properties

◇ A Deza graph has diameter at least 2.

◇ Any Deza graph with parameters (n, k, b, a) is a strongly regular graph with parameters (n, k, λ, μ) if and only if $M = A$, $M = B$, or $b = a$. Then $\{\lambda, \mu\} = \{a, b\}$ and $M^2 = kI + \lambda M + \mu(J - I - M)$.



Petersen graph is $SRG(10, 3, 0, 1)$ and a Deza graph with the same parameters

Deza graphs: properties

◇ A Deza graph of diameter 2 which is not a strongly regular graph is called a *strictly Deza graph*.

Strictly Deza graph with parameters $(8, 4, 2, 1)$
`https://conferences.famnit.upr.si/event/13`

Catalogue: `http://alg.imm.uran.ru/dezagraphs/dezatab.html`

Deza graphs: children

Definition 2

Let G be a Deza graph with parameters (n, k, b, a) and $b \neq a$. The children G_A and G_B of G are defined as two graphs on the same vertex set $V(G)$ such that for any two distinct vertices $u, v \in V(G)$:

- u, v are adjacent in G_A if and only if the number of their common neighbours is equal to a;
- u, v are adjacent in G_B if and only if the number of their common neighbours is equal to b.

G with parameters $(8, 4, 2, 1) \Rightarrow G_A \cong C_4 \bigcup C_4$, $G_B \cong K_8 \backslash G_A$

Deza graphs and their children: spectrum

Theorem. [Akbari-Ghodrati-Hosseinzadeh-Kabanov-Konstantinova-Shalaginov, Spectra of Deza graphs, *Linear and Multilinear Algebra*, 2020]

Let G be a Deza graph with parameters (n, k, b, a), $b > a$. Let M, A, B be the adjacency matrices of G and its children, respectively. If $\theta_1 = k, \theta_2, \dots, \theta_n$ are the eigenvalues of M, then

(i) the eigenvalues of A are

$$\alpha = \frac{b(n-1) - k(k-1)}{b-a}, \ \alpha_2 = \frac{k - b - \theta_2^2}{b-a}, \ \dots, \ \alpha_n = \frac{k - b - \theta_n^2}{b-a}.$$

(ii) the eigenvalues of B are

$$\beta = \frac{a(n-1) - k(k-1)}{a-b}, \ \beta_2 = \frac{k - a - \theta_2^2}{a-b}, \ \dots, \ \beta_n = \frac{k - a - \theta_n^2}{a-b}.$$

Deza graphs and their children: example

Remark

If $b \neq a$, then children G_A and G_B are regular graphs with degrees $\alpha = \dfrac{b(n-1)-k(k-1)}{b-a}$ and $\beta = \dfrac{a(n-1)-k(k-1)}{a-b}$, respectively.

G with parameters $(8,4,2,1) \Rightarrow G_A \cong C_4 \bigcup C_4,\ G_B \cong K_8 \backslash G_A$

$$\alpha = \frac{2\cdot 7 - 4\cdot 3}{2-1} = 2 \text{ and } \beta = \frac{1\cdot 7 - 4\cdot 3}{1-2} = 5.$$

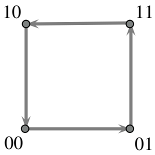
Open spectral question

The question

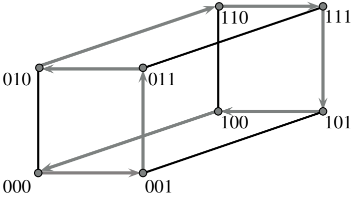
Is a spectrum of a Deza graph defined by its parameters?

Example

The hypercube graph H_n is a Deza graph with parameters $(2^n, n, 2, 0)$. Its spectrum is defined by eigenvalues $n - 2k$ with the multiplicities $\binom{n}{k}$, where $0 \leqslant k \leqslant n$.



H_2



H_3

Resume on Deza graphs: generalization of *SRG*

From structural point of view:

A Deza graph is a generalization of a strongly regular graph such that: the number of common neighbours of any pair of distinct vertices in a Deza graph does not depend on the adjacency.

From spectral point of view:

- Any strongly regular graph has exactly three distinct eigenvalues.
- A Deza graph can have more than three distinct eigenvalues.

Further generalizations

A *strongly Deza graph* is a Deza graph G with parameters (n, k, b, a) whose children are strongly regular graphs.

Strongly Deza graphs

Definition

A *strongly Deza graph* is a Deza graph G with parameters (n, k, b, a) whose children are strongly regular graphs.

First considering, 2021

V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Generalised dual Seidel switching and Deza graphs with strongly regular children, *Discrete Mathematics*, 344(3) (2021) 112238.
`https://doi.org/10.1016/j.disc.2020.112238`

Definition and spectral characterization, 2021+

S. Akbari, W. H. Haemers, M. A. Hosseinzadeh, V. V. Kabanov, E. V. Konstantinova, L. Shalaginov, Spectra of strongly Deza graphs, *Discrete Mathematics*, 344(12) (2021) 112238.
`https://doi.org/10.1016/j.disc.2021.112622`

Integral strongly Deza graphs

Integral graph

A graph Γ is *integral* if its spectrum consists entirely of integers.

F. Harary and A. J. Schwenk, Which graphs have integral spectra? *Graphs and Combinatorics* (1974).

The problem of characterizing integral graphs.

O. Ahmadi, N. Alon, I. F. Blake, and I. E. Shparlinski, Graphs with integral spectrum, (2009)

Most graphs have nonintegral eigenvalues, more precisely, it was proved that the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for a sufficiently large n.

Corollary (AHHKKS-2021+)

The children of a strongly Deza graph are integral graphs.

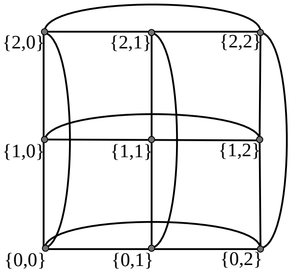
Part III

Neumaier graphs

Strongly regular graph: revisited

Definition

G is a strongly regular graph if:
- every two adjacent vertices have λ common neighbours (edge-regular graph)
- every two non-adjacent vertices have μ common neighbours (co-edge-regular).

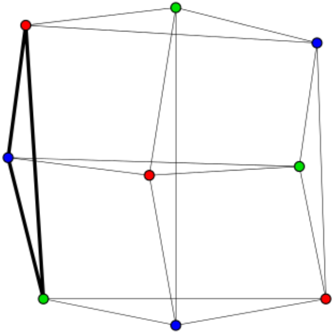


$SRG(9,4,1,2)$: the lattice graph $L_{3,3}$ / 3×3 rook's graph

Regularity of subsets

Definition

A vertex subset $X \subset V$ of a graph $G = (V, E)$ is e-regular if for every $v \notin X$: $|N(v) \cap X| = e$.



3×3 rook's graph is the graph of triangular duoprism

1-regular subset $\cong$ 1-regular clique

Neumaier's question

Theorem (Neumaier, 1981)

A vertex- and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier)

Is a regular, edge-regular graph with a regular clique necessarily SRG?

(strictly) Neumaier graph

A Neumaier graph is a regular, edge-regular graph with a regular clique. It is a strictly Neumaier graph if it is not strongly regular. A Neumaier graph has parameters $(n, k, \lambda; e, s)$ if it is an edge-regular graph with parameters (n, k, λ), admitting an e-regular clique of size s.

Open questions

Do strictly Neumaier graphs exist?
For which parameter sets do strictly Neumaier graphs exist?

Known results

Greaves-Koolen (2018) Edge-regular graphs with regular cliques

There are (infinitely many) strictly Neumaier graphs.

Parametrized Cayley graphs.

Evans-Goryainov-Panasenko (2019)

1. The smallest strictly Neumaier graph.
2. There is an infinite class of strictly Neumaier graphs.

Based on affine polar graphs.

Abaid-De Boeck-Koolen-(2020-2021+)

1. An infinite class of Neumaier graphs and non-existence results.
2. Neumaier graphs with few eigenvalues.

Evans-Gorayinov machine

Let $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$ be edge-regular graphs with parameters (n, k, λ) that admit a partition into perfect 1-codes of size a, where a is a proper divisor of $\lambda + 2$, and $t = \frac{\lambda+2}{a}$. For any $j \in \{1, \ldots, t\}$, let $H_1^{(j)}, \ldots, H_{\frac{n}{a}}^{(j)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(j)}$. Let $\Pi = (\pi_2, \ldots, \pi_t)$ be a $(t-1)$-tuple of permutations from $Sym(\{1, \ldots, \frac{n}{a}\})$.

Denote by $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ the graph obtained as follows.

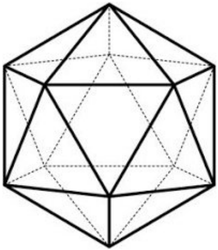
1. Take the disjoint union of the graphs $\Gamma^{(1)}, \ldots, \Gamma^{(t)}$.
2. For any $i \in \{1, \ldots, \frac{n}{a}\}$, connect any two vertices from $H_i^{(1)} \cup H_{\pi_2(i)}^{(2)} \cup \ldots \cup H_{\pi_t(i)}^{(t)}$ to form a 1-regular clique of size at.

Main result

The graph $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ is a Neumaier graph with parameters $(nt, k + at - 1, \lambda; 1, at)$ whose vertex set admits a partition into 1-regular cliques of size at. Moreover, if $t \geqslant 2$, then $F_\Pi(\Gamma^{(1)}, \ldots, \Gamma^{(t)})$ is a strictly Neumaier graph.

Examples

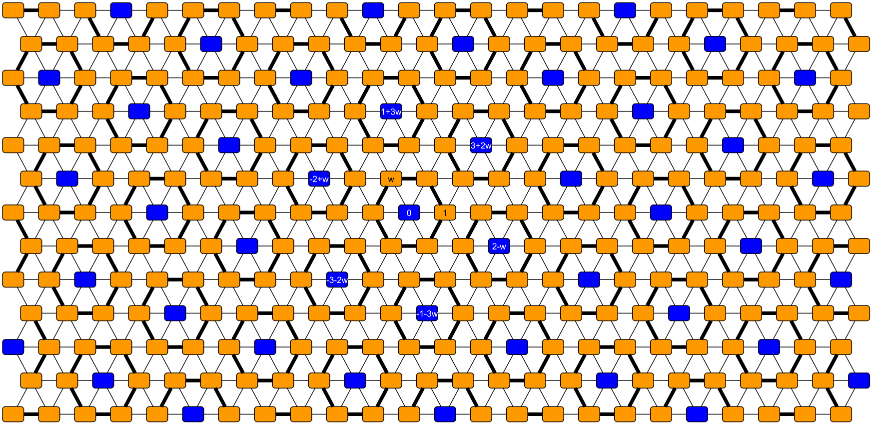
Construction given by a pair of icosahedrons:



The icosahedral graph is an edge-regular graph with parameters $(12, 5, 2)$ that admits a partition into 6 perfect 1-codes of size $a = 2$. Thus, we can use $t = \frac{\lambda+2}{a} = 2$ copies of the icosahedral graph in the general construction to produce four pairwise non-isomorphic strictly Neumaier graphs (depending on the choice of the permutation π_2) with parameters $(24, 8, 2; 1, 4)$.

Construction given by the 6-regular triangular grid

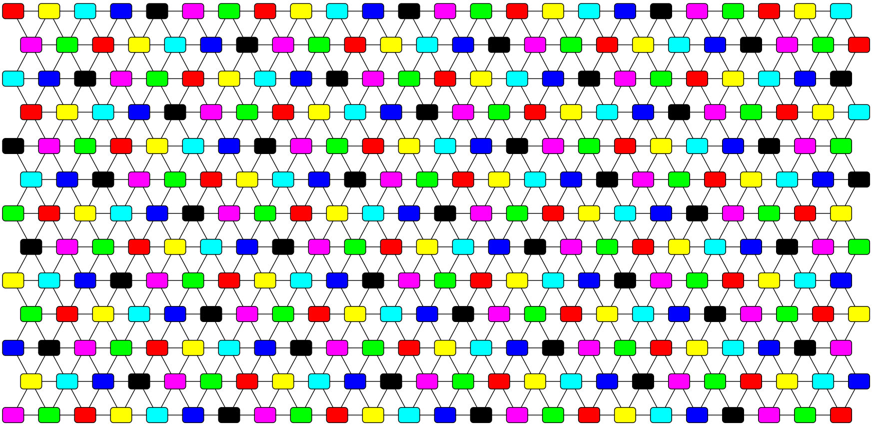
Step 1: find a perfect 1-code



Hint: The ideal I generated by an element of norm 7

Construction given by the 6-regular triangular grid

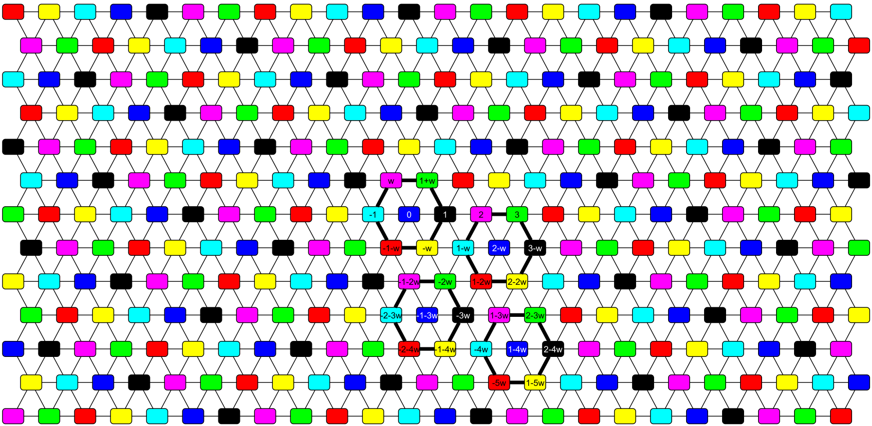
Step 2: find a partition of the triangular grid into 7 perfect 1-codes



Hint: I is an additive subgroup of index 7 in $\mathbb{Z}[\omega]$

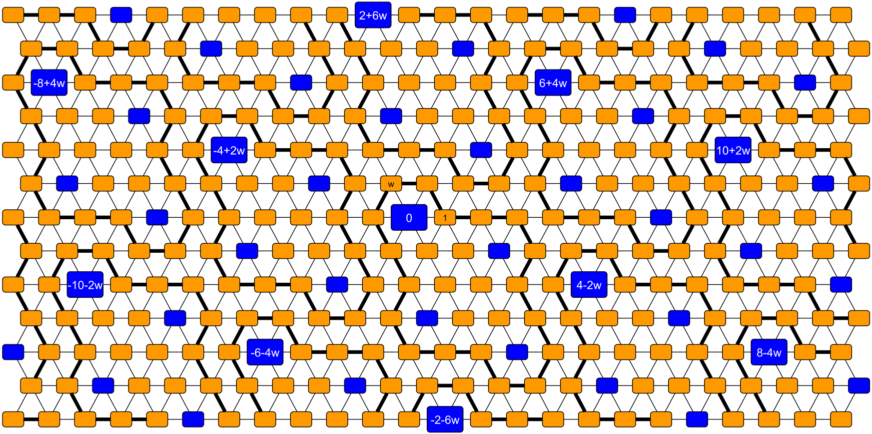
Construction given by the 6-regular triangular grid

Step 3: fix a block of 4 balls of radius 1



Construction given by the 6-regular triangular grid

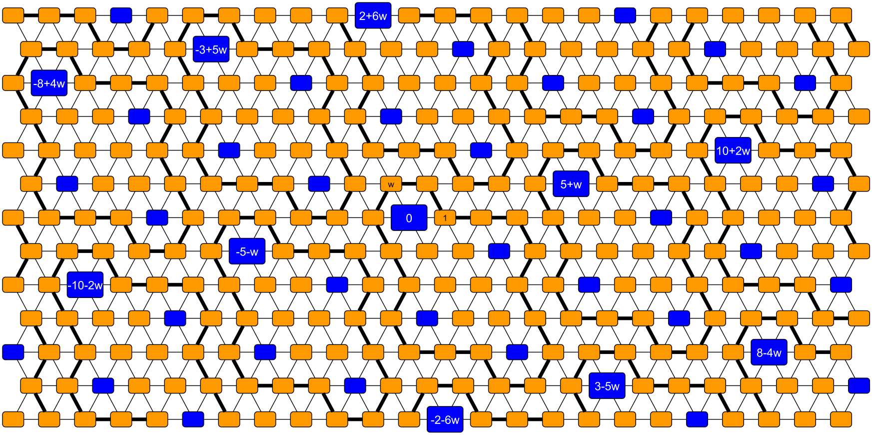
Step 4 – 1: consider a tessellation given an additive subgroup



Hint: additive shifts by $T_1 := \{2(-2+\omega)x + 14y \mid x, y \in \mathbb{Z}\}$

Construction given by the 6-regular triangular grid

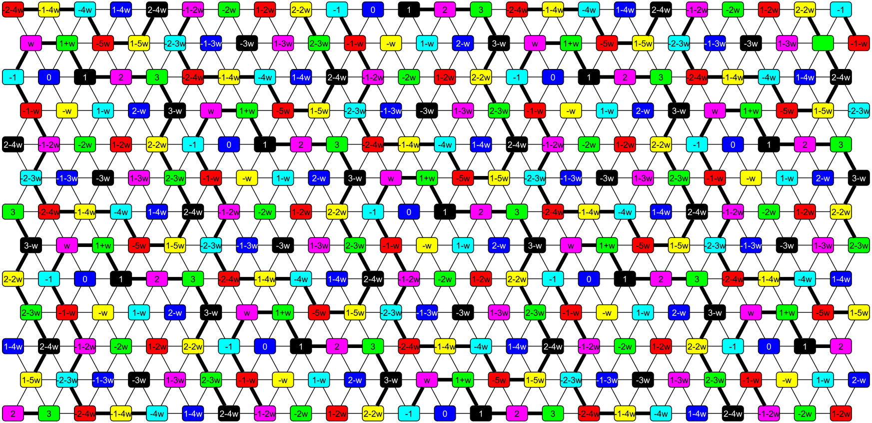
Step 4 – 2: consider a tessellation given an additive subgroup



Hint: additive shifts by $T_2 := \{(5+\omega)x + 28y \mid x, y \in \mathbb{Z}\}$

Construction given by the 6-regular triangular grid

Step 5 – 1: consider a quotient graph Δ_1 of the triangular grid by T_1

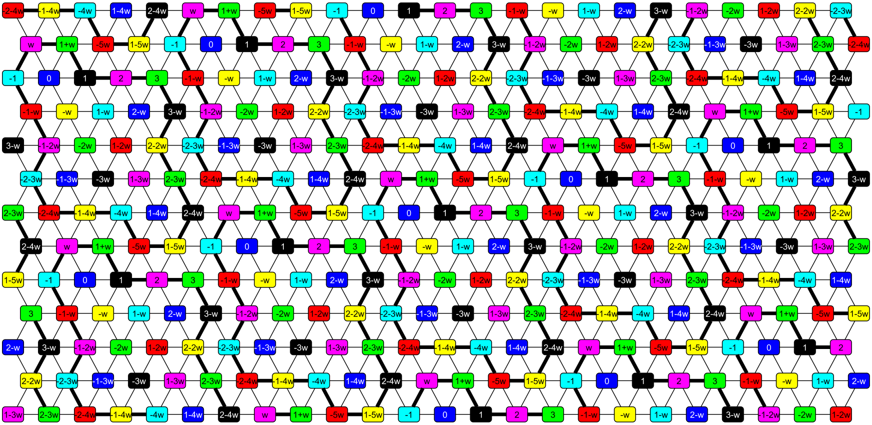


Hint: $\Delta_1 := Cay(G_1, \{\pm(1+T_1), \pm(\omega+T_1), \pm(\omega^2+T_1)\})$, where

$$G_1 := \mathbb{Z}[\omega]/T_1$$

Construction given by the 6-regular triangular grid

Step 5 – 2: consider a quotient graph Δ_2 of the triangular grid by T_2



Hint: $\Delta_2 := Cay(G_2, \{\pm(1+T_2), \pm(\omega+T_2), \pm(\omega^2+T_2)\})$, where
$G_2 := \mathbb{Z}[\omega]/T_2$

Construction given by the 6-regular triangular grid

Finally,

• each of the graphs Δ_1 and Δ_2 is edge-regular with parameters $(28, 6, 2)$

• and admits a partition into perfect 1-codes of size $a = 4$;

• these partitions are given by the original partition of the triangular grid into perfect 1-codes;

• apply Evans-Goryainov machine and get two strictly Neumaier graphs with parameters $(28, 9, 2; 1, 4)$.

https://arxiv.org/abs/2109.13884:

A general construction of strictly Neumaier graphs and related switching, by Rhys J. Evans, Sergey Goryainov, Elena V. Konstantinova, Alexander D. Mednykh

Further questions

- n-dimensional case of the triangular grid?
- other grids? (operation on grids?)
- how one can use root systems?
- generalization to hyperbolic spaces?

Main problems:

◇ find a perfect code

◇ find a subgroup