Факторно делимые абелевы группы, E-кольца и связи между ними

Е. А. Тимошенко, М. Н. Зонов

Томский государственный университет

Декабрьские чтения — 2021

Quotient divisible groups

Definition [R. A. Beaumont, R. S. Pierce, 1961].

Quotient divisible groups

Definition [R. A. Beaumont, R. S. Pierce, 1961].
A torsion-free group G is *quotient divisible* (or simply *q.d.*) if there is a finite-rank free subgroup $F \subset G$ such that G/F is a direct sum of a divisible torsion group and a bounded group.

Quotient divisible groups

Definition [R. A. Beaumont, R. S. Pierce, 1961].
A torsion-free group G is *quotient divisible* (or simply *q.d.*) if there is a finite-rank free subgroup $F \subset G$ such that G/F is a direct sum of a divisible torsion group and a bounded group.

Theorem 1 [BP, 1961]. *The following are equivalent:*

1) *G is a q.d. group.*

2) *There is a finite-rank free subgroup $F \subset G$ such that G/F is a divisible torsion group.*

Quotient divisible groups

Definition [R. A. Beaumont, R. S. Pierce, 1961].
A torsion-free group G is *quotient divisible* (or simply *q.d.*) if there is a finite-rank free subgroup $F \subset G$ such that G/F is a direct sum of a divisible torsion group and a bounded group.

Theorem 1 [BP, 1961]. *The following are equivalent:*

1) *G is a q.d. group.*

2) *There is a finite-rank free subgroup $F \subset G$ such that G/F is a divisible torsion group.*

In 1998 Fomin and Wickless extended this definition to arbitrary Abelian groups.

Definition [A. A. Fomin, W. Wickless, 1998].
Let $n \geqslant 0$. A group G is a *q.d. group of rank* n if its torsion part $T(G)$ is reduced and there is a free subgroup $F \subset G$ of rank n such that G/F is a divisible torsion group.

Every such subgroup F is a *fundamental subgroup* of G, and any free basis of a fundamental subgroup F is a *fundamental system* of elements of G.

Definition [A. A. Fomin, W. Wickless, 1998].
Let $n \geqslant 0$. A group G is a *q.d. group of rank* n if its torsion part $T(G)$ is reduced and there is a free subgroup $F \subset G$ of rank n such that G/F is a divisible torsion group.

Every such subgroup F is a *fundamental subgroup* of G, and any free basis of a fundamental subgroup F is a *fundamental system* of elements of G.

In 2007 Davydova described all q.d. groups of rank 1.

Definition [A. A. Fomin, W. Wickless, 1998].
Let $n \geqslant 0$. A group G is a *q.d. group of rank* n if its torsion part $T(G)$ is reduced and there is a free subgroup $F \subset G$ of rank n such that G/F is a divisible torsion group.

Every such subgroup F is a *fundamental subgroup* of G, and any free basis of a fundamental subgroup F is a *fundamental system* of elements of G.

In 2007 Davydova described all q.d. groups of rank 1.

Let P be the set of all primes.

Definition [A. A. Fomin, W. Wickless, 1998].
Let $n \geqslant 0$. A group G is a *q.d. group of rank* n if its torsion part $T(G)$ is reduced and there is a free subgroup $F \subset G$ of rank n such that G/F is a divisible torsion group.

Every such subgroup F is a *fundamental subgroup* of G, and any free basis of a fundamental subgroup F is a *fundamental system* of elements of G.

In 2007 Davydova described all q.d. groups of rank 1.

Let P be the set of all primes.

Theorem 2 [O. I. Davydova, 2007].
a) *For any characteristic* $\chi = (n_p)_{p \in P}$, *the additive group of* R^χ *is a q.d. group of rank* 1 *with fundamental subgroup* $\langle 1 \rangle$.

Definition [A. A. Fomin, W. Wickless, 1998].
Let $n \geqslant 0$. A group G is a *q.d. group of rank* n if its torsion part $T(G)$ is reduced and there is a free subgroup $F \subset G$ of rank n such that G/F is a divisible torsion group.

Every such subgroup F is a *fundamental subgroup* of G, and any free basis of a fundamental subgroup F is a *fundamental system* of elements of G.

In 2007 Davydova described all q.d. groups of rank 1.

Let P be the set of all primes.

Theorem 2 [O. I. Davydova, 2007].

a) *For any characteristic* $\chi = (n_p)_{p \in P}$, *the additive group of* R^χ *is a q.d. group of rank* 1 *with fundamental subgroup* $\langle 1 \rangle$.

b) *Any q.d. group of rank* 1 *is isomorphic to the additive group of some* R^χ.

Rings $\boldsymbol{R^\chi}$

$$\chi = (n_p)_{p \in P} \text{ with } n_p \in \mathbb{N} \cup \{0, \infty\};$$

$$L = \left\{ p \in P \;\middle|\; 0 < n_p < \infty \right\};$$

$$K^\chi = \prod_{p \in L} \mathbb{Z}/p^{n_p}\mathbb{Z};$$

Rings R^χ

$$\chi = (n_p)_{p \in P} \text{ with } n_p \in \mathbb{N} \cup \{0, \infty\};$$

$$L = \left\{ p \in P \;\middle|\; 0 < n_p < \infty \right\};$$

$$K^\chi = \prod_{p \in L} \mathbb{Z}/p^{n_p}\mathbb{Z};$$

$\mathbb{Q}^\chi$ is the subring of the field $\mathbb{Q}$ generated by the elements $\frac{1}{p}$ such that $n_p < \infty$.

Rings $\boldsymbol{R^\chi}$

$$\chi = (n_p)_{p\in P} \text{ with } n_p \in \mathbb{N}\cup\{0,\infty\};$$
$$L = \left\{p \in P \;\middle|\; 0 < n_p < \infty\right\};$$
$$K^\chi = \prod_{p\in L} \mathbb{Z}/p^{n_p}\mathbb{Z};$$

$\mathbb{Q}^\chi$ is the subring of the field $\mathbb{Q}$ generated by the elements $\frac{1}{p}$ such that $n_p < \infty$.

If L is a finite set, then $R^\chi = \mathbb{Q}^\chi \times K^\chi$.

Rings R^χ

$$\chi = (n_p)_{p\in P} \text{ with } n_p \in \mathbb{N}\cup\{0,\infty\};$$
$$L = \left\{p \in P \;\middle|\; 0 < n_p < \infty\right\};$$
$$K^\chi = \prod_{p\in L} \mathbb{Z}/p^{n_p}\mathbb{Z};$$

$\mathbb{Q}^\chi$ is the subring of the field $\mathbb{Q}$ generated by the elements $\frac{1}{p}$ such that $n_p < \infty$.

If L is a finite set, then $R^\chi = \mathbb{Q}^\chi \times K^\chi$.

If L is infinite, then R^χ is the ring of all elements $b = (b_p)_{p\in L} \in K^\chi$ such that for some fraction $\frac{u}{v} \in \mathbb{Q}^\chi$ the equality $ue_p = vb_p$ (where e_p is the identity of the ring $\mathbb{Z}/p^{n_p}\mathbb{Z}$) holds for almost all $p \in L$.

Torsion-free finite-rank groups and q.d. groups

Torsion-free finite-rank groups and q.d. groups

Theorem 3 [FW, 1998].
The category of torsion-free finite-rank groups (with quasi-homomorphisms as morphisms) is dual to the category of q.d. groups (with quasi-homomorphisms as morphisms).

Torsion-free finite-rank groups and q.d. groups

Theorem 3 [FW, 1998].
The category of torsion-free finite-rank groups (*with quasi-homomorphisms as morphisms*) *is dual to the category of q.d. groups* (*with quasi-homomorphisms as morphisms*).

A *quasi-homomorphism* is an element of the group $\mathbb{Q} \otimes \mathrm{Hom}(A, B)$.

Two groups (rings) A and B are *quasi-isomorphic* if there exist subgroups (subrings) $A' \subset A$ and $B' \subset B$ such that $nA \subset A'$, $nB \subset B'$ (for some $n \in \mathbb{N}$) and $A' \cong B'$.

Definition. For $p \in P$, the *p-rank* of G is the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ [notation: $r_p(G)$].

Definition. For $p \in P$, the *p-rank* of G is the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ [notation: $r_p(G)$].

Proposition 4. *Let G be a q.d. group. Then*

a) *every fundamental system of G contains at least $r_p(G)$ elements which are not in pG;*

Definition. For $p \in P$, the *p-rank* of G is the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ [notation: $r_p(G)$].

Proposition 4. *Let G be a q.d. group. Then*

a) *every fundamental system of G contains at least $r_p(G)$ elements which are not in pG;*

b) *every fundamental subgroup of G has a free basis with exactly $r_p(G)$ elements which are not in pG.*

Definition. For $p \in P$, the *p-rank* of G is the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ [notation: $r_p(G)$].

Proposition 4. *Let G be a q.d. group. Then*

a) *every fundamental system of G contains at least $r_p(G)$ elements which are not in pG;*

b) *every fundamental subgroup of G has a free basis with exactly $r_p(G)$ elements which are not in pG.*

Theorem 5. *Let G be a q.d. group of rank n and F be a fundamental subgroup of G. Then*

$$G/F \cong \bigoplus_{p \in P} \bigoplus_{n - n_p} \mathbb{Z}(p^\infty),$$

where $n_p = r_p(G/T(G))$.

Let G be a torsion-free group of rank n and $g_1, g_2, \ldots, g_n$ be a system of independent elements.
It is known that the type $\mathbf{t}(g_1) \wedge \mathbf{t}(g_2) \wedge \ldots \wedge \mathbf{t}(g_n)$ does not depend on the choice of $g_1, g_2, \ldots, g_n$.

Let G be a torsion-free group of rank n and $g_1, g_2, \ldots, g_n$ be a system of independent elements.
It is known that the type $\mathbf{t}(g_1) \wedge \mathbf{t}(g_2) \wedge \ldots \wedge \mathbf{t}(g_n)$ does not depend on the choice of $g_1, g_2, \ldots, g_n$.

This type is denoted by $\mathbf{it}(G)$ [the *inner type* of G].

Let G be a torsion-free group of rank n and $g_1, g_2, \ldots, g_n$ be a system of independent elements.

It is known that the type $\mathbf{t}(g_1) \wedge \mathbf{t}(g_2) \wedge \ldots \wedge \mathbf{t}(g_n)$ does not depend on the choice of $g_1, g_2, \ldots, g_n$.

This type is denoted by $\mathbf{it}(G)$ [the *inner type* of G].

Proposition 6. *If G is a torsion-free q.d. group, then the type $\mathbf{it}(G)$ is idempotent.*

Let G be a torsion-free group of rank n and $g_1, g_2, \dots, g_n$ be a system of independent elements.
It is known that the type $\mathbf{t}(g_1) \wedge \mathbf{t}(g_2) \wedge \dots \wedge \mathbf{t}(g_n)$ does not depend on the choice of $g_1, g_2, \dots, g_n$.
This type is denoted by $\mathbf{it}(G)$ [the *inner type* of G].

Proposition 6. *If G is a torsion-free q.d. group, then the type $\mathbf{it}(G)$ is idempotent.*

Definition. Let H be a group with $T(H)$ reduced and F be a finite-rank free subgroup of H.
The sum of all q.d. subgroups $G \subset H$ such that F is a fundamental subgroup of G is called the *quotient divisible hull* of F in H.

Let G be a torsion-free group of rank n and $g_1, g_2, \ldots, g_n$ be a system of independent elements.
It is known that the type $\mathbf{t}(g_1) \wedge \mathbf{t}(g_2) \wedge \ldots \wedge \mathbf{t}(g_n)$ does not depend on the choice of $g_1, g_2, \ldots, g_n$.
This type is denoted by $\mathbf{it}(G)$ [the *inner type* of G].

Proposition 6. *If G is a torsion-free q.d. group, then the type $\mathbf{it}(G)$ is idempotent.*

Definition. Let H be a group with $T(H)$ reduced and F be a finite-rank free subgroup of H.
The sum of all q.d. subgroups $G \subset H$ such that F is a fundamental subgroup of G is called the *quotient divisible hull* of F in H.

Theorem 7. *The q.d. hull of F in H is the largest q.d. subgroup of H that has F as its fundamental subgroup.*

Theorem 8. *Let F be a finite-rank free subgroup of H. The following are equivalent:*

Theorem 8. *Let F be a finite-rank free subgroup of H. The following are equivalent:*

1) *H is a q.d. group which has F as its fundamental subgroup.*

2) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \ldots G_n \subset \ldots$ which have F as their fundamental subgroup.*

3) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \ldots G_n \subset \ldots$ which have F as their fundamental subgroup and have finite torsion parts.*

Theorem 8. *Let F be a finite-rank free subgroup of H. The following are equivalent:*

1) *H is a q.d. group which has F as its fundamental subgroup.*

2) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \dots G_n \subset \dots$ which have F as their fundamental subgroup.*

3) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \dots G_n \subset \dots$ which have F as their fundamental subgroup and have finite torsion parts.*

Theorem 9 [A. A. Fomin].
The class of q.d. groups is closed under extensions.

Theorem 8. *Let F be a finite-rank free subgroup of H. The following are equivalent:*

1) *H is a q.d. group which has F as its fundamental subgroup.*

2) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \ldots G_n \subset \ldots$ which have F as their fundamental subgroup.*

3) *H is the union of a chain of q.d. subgroups $G_1 \subset G_2 \subset \ldots G_n \subset \ldots$ which have F as their fundamental subgroup and have finite torsion parts.*

Theorem 9 [A. A. Fomin].
The class of q.d. groups is closed under extensions.

Theorem 10. *For any q.d. group H, there is a chain of subgroups $0 = G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n = H$ such that all G_{i+1}/G_i are q.d. groups of rank 1.*

Quotient divisible groups of rank two

For any isomorphism $\varphi\colon A/A_0 \to B/B_0$, we can consider the group $\left\{(a,b) \in A \oplus B \;\middle|\; \varphi(a + A_0) = b + B_0\right\}$.

Quotient divisible groups of rank two

For any isomorphism $\varphi\colon A/A_0 \to B/B_0$, we can consider the group $\left\{(a,b) \in A \oplus B \;\middle|\; \varphi(a + A_0) = b + B_0\right\}$.

Any torsion-free group G of rank 2 can be embedded in $\mathbb{Q} \oplus \mathbb{Q}$, so G has a representation of the following form:

Quotient divisible groups of rank two

For any isomorphism $\varphi\colon A/A_0 \to B/B_0$, we can consider the group $\big\{(a,b) \in A \oplus B \bigm| \varphi(a + A_0) = b + B_0\big\}$.

Any torsion-free group G of rank 2 can be embedded in $\mathbb{Q} \oplus \mathbb{Q}$, so G has a representation of the following form:

$$G \cong H = \big\{(a,b) \in A \oplus B \bigm| \varphi(a + A_0) = b + B_0\big\}$$
$$(0 \neq A_0 \subset A \subset \mathbb{Q}, \quad 0 \neq B_0 \subset B \subset \mathbb{Q}, \quad A/A_0 \overset{\varphi}{\cong} B/B_0). \tag{1}$$

Quotient divisible groups of rank two

For any isomorphism $\varphi\colon A/A_0 \to B/B_0$, we can consider the group $\{(a,b) \in A \oplus B \mid \varphi(a + A_0) = b + B_0\}$.

Any torsion-free group G of rank 2 can be embedded in $\mathbb{Q} \oplus \mathbb{Q}$, so G has a representation of the following form:

$$\begin{gathered} G \cong H = \{(a,b) \in A \oplus B \mid \varphi(a + A_0) = b + B_0\} \\ (0 \neq A_0 \subset A \subset \mathbb{Q}, \quad 0 \neq B_0 \subset B \subset \mathbb{Q}, \quad A/A_0 \overset{\varphi}{\cong} B/B_0). \end{gathered} \tag{1}$$

Such a representation is said to be

- *good* if the types $\mathbf{t}(A)$, $\mathbf{t}(B)$ and $\mathbf{t}(A_0) \wedge \mathbf{t}(B_0)$ are idempotent;

Quotient divisible groups of rank two

For any isomorphism $\varphi\colon A/A_0 \to B/B_0$, we can consider the group $\{(a,b) \in A \oplus B \mid \varphi(a+A_0) = b + B_0\}$.

Any torsion-free group G of rank 2 can be embedded in $\mathbb{Q} \oplus \mathbb{Q}$, so G has a representation of the following form:

$$\begin{gathered} G \cong H = \{(a,b) \in A \oplus B \mid \varphi(a+A_0) = b + B_0\} \\ (0 \neq A_0 \subset A \subset \mathbb{Q}, \quad 0 \neq B_0 \subset B \subset \mathbb{Q}, \quad A/A_0 \overset{\varphi}{\cong} B/B_0). \end{gathered} \tag{1}$$

Such a representation is said to be

- *good* if the types $\mathbf{t}(A)$, $\mathbf{t}(B)$ and $\mathbf{t}(A_0) \wedge \mathbf{t}(B_0)$ are idempotent;
- *very good* if it is good and the group $A/A_0 \cong B/B_0$ is divisible.

Theorem 11. *For a torsion-free group G of rank 2, the following are equivalent:*

1) *G is a q.d. group.*

2) *G has a good representation of the form (1).*

3) *G has a very good representation of the form (1).*

4) *Every representation of G of the form (1) is good.*

5) *The type $\mathbf{it}(G)$ is idempotent, and every rank-1 torsion-free homomorphic image of G has an idempotent type.*

Theorem 11. *For a torsion-free group G of rank 2, the following are equivalent:*

1) *G is a q.d. group.*

2) *G has a good representation of the form* (1).

3) *G has a very good representation of the form* (1).

4) *Every representation of G of the form* (1) *is good.*

5) *The type $\mathbf{it}(G)$ is idempotent, and every rank-1 torsion-free homomorphic image of G has an idempotent type.*

Example 12. There is a torsion-free q.d. group G of rank 2 such that the type $\mathbf{t}(x)$ is nonidempotent for every $x \in G \setminus \{0\}$.

Theorem 11. *For a torsion-free group G of rank 2, the following are equivalent:*

1) *G is a q.d. group.*

2) *G has a good representation of the form* (1).

3) *G has a very good representation of the form* (1).

4) *Every representation of G of the form* (1) *is good.*

5) *The type* $\mathbf{it}(G)$ *is idempotent, and every rank-*1 *torsion-free homomorphic image of G has an idempotent type.*

Example 12. There is a torsion-free q.d. group G of rank 2 such that the type $\mathbf{t}(x)$ is nonidempotent for every $x \in G \setminus \{0\}$.

Remark. In particular, G can not be endowed with a ring structure.

Example 13. Let $L = \left\{ p \in P \;\middle|\; p = 4k+1 \right\}$ and $K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}$.

Example 13. Let $L = \left\{ p \in P \;\middle|\; p = 4k+1 \right\}$ and $K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}$.

If G is the pure hull of $\{1, b\}$ in K, where $b^2 = -1$, then

- G is a mixed q.d. group of rank 2;

Example 13. Let $L = \{p \in P \mid p = 4k+1\}$ and $K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}$.

If G is the pure hull of $\{1, b\}$ in K, where $b^2 = -1$, then

- G is a mixed q.d. group of rank 2;
- G does not have a representation of the form (1) such that A and B are q.d. groups of rank 1.

Example 13. Let $L = \{p \in P \mid p = 4k+1\}$ and
$$K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}.$$

If G is the pure hull of $\{1, b\}$ in K, where $b^2 = -1$, then

- G is a mixed q.d. group of rank 2;
- G does not have a representation of the form (1) such that A and B are q.d. groups of rank 1.

Definition. A q.d. group G is *p-minimal* if $G/F \cong \mathbb{Z}(p^\infty)$ for every fundamental subgroup F of G.

Example 13. Let $L = \{p \in P \mid p = 4k+1\}$ and $K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}$.

If G is the pure hull of $\{1, b\}$ in K, where $b^2 = -1$, then

- G is a mixed q.d. group of rank 2;
- G does not have a representation of the form (1) such that A and B are q.d. groups of rank 1.

Definition. A q.d. group G is *p-minimal* if $G/F \cong \mathbb{Z}(p^\infty)$ for every fundamental subgroup F of G.

Remark. A torsion-free q.d. group G of rank n is p-minimal if and only if $r_p(G) = n-1$ and $r_q(G) = n$ for all $q \in P \setminus \{p\}$.

Example 13. Let $L = \{p \in P \mid p = 4k+1\}$ and $K = \prod_{p \in L} \mathbb{Z}/p\mathbb{Z}$.

If G is the pure hull of $\{1, b\}$ in K, where $b^2 = -1$, then

- G is a mixed q.d. group of rank 2;
- G does not have a representation of the form (1) such that A and B are q.d. groups of rank 1.

Definition. A q.d. group G is *p-minimal* if $G/F \cong \mathbb{Z}(p^\infty)$ for every fundamental subgroup F of G.

Remark. A torsion-free q.d. group G of rank n is p-minimal if and only if $r_p(G) = n-1$ and $r_q(G) = n$ for all $q \in P \setminus \{p\}$.

For some results concerning torsion-free p-minimal q.d. groups and their endomorphism rings see [Fomin, 1984].

Torsion-free p-minimal q.d. groups of rank 2

J_p is the ring of p-adic integers;
$U(J_p)$ is the multiplicative group of J_p;
$\mathbb{Q}^{(p)}$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{p}$.

Torsion-free p-minimal q.d. groups of rank 2

J_p is the ring of p-adic integers;
$U(J_p)$ is the multiplicative group of J_p;
$\mathbb{Q}^{(p)}$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{p}$.

Theorem 14. *For a group G, the following are equivalent:*

1) *G is a torsion-free p-minimal q.d. group of rank 2.*
2) *There is $\eta \in U(J_p)$ such that $G \cong H_\eta$, where*

$$H_\eta = \left\{ (a,b) \in \mathbb{Q}^{(p)} \oplus \mathbb{Q}^{(p)} \;\middle|\; \eta(a+\mathbb{Z}) = b + \mathbb{Z} \right\}.$$

Torsion-free p-minimal q.d. groups of rank 2

J_p is the ring of p-adic integers;
$U(J_p)$ is the multiplicative group of J_p;
$\mathbb{Q}^{(p)}$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{p}$.

Theorem 14. *For a group G, the following are equivalent:*

1) *G is a torsion-free p-minimal q.d. group of rank 2.*
2) *There is $\eta \in U(J_p)$ such that $G \cong H_\eta$, where*

$$H_\eta = \left\{ (a, b) \in \mathbb{Q}^{(p)} \oplus \mathbb{Q}^{(p)} \;\middle|\; \eta(a + \mathbb{Z}) = b + \mathbb{Z} \right\}.$$

Theorem 15. *For $\eta \in U(J_p)$, the following are equivalent:*

1) *η is rational.*
2) *H_η is a completely decomposable group.*
3) *$H_\eta \cong \mathbb{Q}^{(p)} \oplus \mathbb{Z}$.*

Torsion-free p-minimal q.d. groups of rank 2

J_p is the ring of p-adic integers;
$U(J_p)$ is the multiplicative group of J_p;
$\mathbb{Q}^{(p)}$ is the subring of $\mathbb{Q}$ generated by $\frac{1}{p}$.

Theorem 14. *For a group G, the following are equivalent:*
1) *G is a torsion-free p-minimal q.d. group of rank 2.*
2) *There is $\eta \in U(J_p)$ such that $G \cong H_\eta$, where*

$$H_\eta = \left\{ (a,b) \in \mathbb{Q}^{(p)} \oplus \mathbb{Q}^{(p)} \;\middle|\; \eta(a+\mathbb{Z}) = b+\mathbb{Z} \right\}.$$

Theorem 15. *For $\eta \in U(J_p)$, the following are equivalent:*
1) *η is rational.*
2) *H_η is a completely decomposable group.*
3) *$H_\eta \cong \mathbb{Q}^{(p)} \oplus \mathbb{Z}$.*

Proposition 16. *If a number $\eta \in U(J_p)$ is not rational, then all rank-1 subgroups of H_η are isomorphic to $\mathbb{Z}$.*

Proposition 17. *If a number $\eta \in U(J_p)$ is not rational, then H_η is isomorphic to the p-pure hull of $\{1, \eta\}$ in J_p.*

Theorem 18. *For $\eta, \zeta \in U(J_p)$, the following are equivalent:*

1) $H_\eta \cong H_\zeta$.

2) *There are $a, b, c, d \in \mathbb{Z}$ such that $\zeta = \dfrac{c + d\eta}{a + b\eta}$ and $ad - bc \in \{\pm 1, \pm p, \pm p^2, \ldots\}$.*

Proposition 17. *If a number $\eta \in U(J_p)$ is not rational, then H_η is isomorphic to the p-pure hull of $\{1, \eta\}$ in J_p.*

Theorem 18. *For $\eta, \zeta \in U(J_p)$, the following are equivalent:*

1) $H_\eta \cong H_\zeta$.

2) *There are $a, b, c, d \in \mathbb{Z}$ such that $\zeta = \dfrac{c + d\eta}{a + b\eta}$ and $ad - bc \in \{\pm 1, \pm p, \pm p^2, \dots\}$.*

Example 19. Choose $\eta \in U(J_p)$ which is not a root of any quadratic polynomial from $\mathbb{Z}[x]$ and a prime $q \neq p$.

Proposition 17. *If a number $\eta \in U(J_p)$ is not rational, then H_η is isomorphic to the p-pure hull of $\{1, \eta\}$ in J_p.*

Theorem 18. *For $\eta, \zeta \in U(J_p)$, the following are equivalent:*

1) $H_\eta \cong H_\zeta$.

2) *There are $a, b, c, d \in \mathbb{Z}$ such that* $\zeta = \dfrac{c + d\eta}{a + b\eta}$ *and* $ad - bc \in \{\pm 1, \pm p, \pm p^2, \ldots\}$.

Example 19. Choose $\eta \in U(J_p)$ which is not a root of any quadratic polynomial from $\mathbb{Z}[x]$ and a prime $q \neq p$.

By Theorem 18, we have $H_\eta \ncong H_{q\eta}$.

Proposition 17. *If a number $\eta \in U(J_p)$ is not rational, then H_η is isomorphic to the p-pure hull of $\{1, \eta\}$ in J_p.*

Theorem 18. *For $\eta, \zeta \in U(J_p)$, the following are equivalent:*

1) $H_\eta \cong H_\zeta$.

2) *There are $a, b, c, d \in \mathbb{Z}$ such that $\zeta = \dfrac{c + d\eta}{a + b\eta}$ and $ad - bc \in \{\pm 1, \pm p, \pm p^2, \ldots\}$.*

Example 19. Choose $\eta \in U(J_p)$ which is not a root of any quadratic polynomial from $\mathbb{Z}[x]$ and a prime $q \neq p$.

By Theorem 18, we have $H_\eta \ncong H_{q\eta}$.

On the other hand, there exist monomorphisms $H_\eta \to H_{q\eta}$ and $H_{q\eta} \to H_\eta$.

$\boldsymbol{E}$-rings

Definition [P. Schultz, 1973]. A ring R is an *E-ring* if every endomorphism of R^+ (the additive group of R) is a left multiplication λ_r by some $r \in R$.
(In this case the correspondence $r \mapsto \lambda_r$ is a ring isomorphism $R \to \operatorname{End} R^+$.)

$\boldsymbol{E}$-rings

Definition [P. Schultz, 1973]. A ring R is an *E-ring* if every endomorphism of R^+ (the additive group of R) is a left multiplication λ_r by some $r \in R$.
(In this case the correspondence $r \mapsto \lambda_r$ is a ring isomorphism $R \to \operatorname{End} R^+$.)

Every E-ring is a commutative ring with identity.

$\boldsymbol{E}$-rings

Definition [P. Schultz, 1973]. A ring R is an *E-ring* if every endomorphism of R^+ (the additive group of R) is a left multiplication λ_r by some $r \in R$.
(In this case the correspondence $r \mapsto \lambda_r$ is a ring isomorphism $R \to \operatorname{End} R^+$.)

Every E-ring is a commutative ring with identity.

Definition. A ring R is a *generalized E-ring* if $R \cong \operatorname{End} R^+$.

E-rings

Definition [P. Schultz, 1973]. A ring R is an *E-ring* if every endomorphism of R^+ (the additive group of R) is a left multiplication λ_r by some $r \in R$.
(In this case the correspondence $r \mapsto \lambda_r$ is a ring isomorphism $R \to \operatorname{End} R^+$.)

Every E-ring is a commutative ring with identity.

Definition. A ring R is a *generalized E-ring* if $R \cong \operatorname{End} R^+$.

Theorem 20 [R. Göbel, S. Shelah, L. Strüngmann, 2004].
There are generalized E-rings which are not E-rings.

By the *rank* of R we mean the torsion-free rank of R^+.

By the *rank* of R we mean the torsion-free rank of R^+.

Proposition 21. *Finite-rank generalized E-rings are E-rings.*

By the *rank* of R we mean the torsion-free rank of R^+.

Proposition 21. *Finite-rank generalized E-rings are E-rings.*

It follows from the results of Bowshell and Schultz [1977] and of Beaumont and Pierce that the additive group of every torsion-free finite-rank E-ring is q.d.

By the *rank* of R we mean the torsion-free rank of R^+.

Proposition 21. *Finite-rank generalized E-rings are E-rings.*

It follows from the results of Bowshell and Schultz [1977] and of Beaumont and Pierce that the additive group of every torsion-free finite-rank E-ring is q.d.

Question [A. V. Tsarev]. Is it true that the additive group of every E-ring of finite rank $\geqslant 1$ is q.d.?

By the *rank* of R we mean the torsion-free rank of R^+.

Proposition 21. *Finite-rank generalized E-rings are E-rings.*

It follows from the results of Bowshell and Schultz [1977] and of Beaumont and Pierce that the additive group of every torsion-free finite-rank E-ring is q.d.

Question [A. V. Tsarev]. Is it true that the additive group of every E-ring of finite rank $\geqslant 1$ is q.d.?

Proposition 22 [BSch, 1977]. *E-rings of rank 0 are exactly the rings $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$ (up to isomorphism).*

By the *rank* of R we mean the torsion-free rank of R^+.

Proposition 21. *Finite-rank generalized E-rings are E-rings.*

It follows from the results of Bowshell and Schultz [1977] and of Beaumont and Pierce that the additive group of every torsion-free finite-rank E-ring is q.d.

Question [A. V. Tsarev]. Is it true that the additive group of every E-ring of finite rank $\geqslant 1$ is q.d.?

Proposition 22 [BSch, 1977]. *E-rings of rank 0 are exactly the rings $\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$ (up to isomorphism).*

Theorem 23 [Ts, 2017]. *E-rings of rank 1 are exactly the rings R^χ (up to isomorphism).*

$\boldsymbol{E}$-ring of rank 2 whose additive group is not q.d.

$\mathbb{Q}^{(L)}$ is the subring of $\mathbb{Q}$ generated by all $\frac{1}{p}$, where $p \in L$.

$\boldsymbol{E}$-ring of rank 2 whose additive group is not q.d.

$\mathbb{Q}^{(L)}$ is the subring of $\mathbb{Q}$ generated by all $\frac{1}{p}$, where $p \in L$.

Example 24. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \qquad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K,$$

$b = (p + p^2\mathbb{Z})_{p \in L} \in K$. Denote

$\boldsymbol{E}$-ring of rank 2 whose additive group is not q.d.

$\mathbb{Q}^{(L)}$ is the subring of $\mathbb{Q}$ generated by all $\frac{1}{p}$, where $p \in L$.

Example 24. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \qquad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K,$$

$b = (p + p^2\mathbb{Z})_{p \in L} \in K$. Denote

$$\overline{R} = (1+T)\mathbb{Q}^{(L)} \oplus (b+T)Y \subset K/T,$$

where $\mathbb{Q}^{(L)} \subset Y \subset \mathbb{Q}$ and the type $\mathbf{t}(Y)$ is nonidempotent.

E-ring of rank 2 whose additive group is not q.d.

$\mathbb{Q}^{(L)}$ is the subring of $\mathbb{Q}$ generated by all $\frac{1}{p}$, where $p \in L$.

Example 24. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \qquad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K,$$

$b = (p + p^2\mathbb{Z})_{p \in L} \in K$. Denote

$$\overline{R} = (1 + T)\mathbb{Q}^{(L)} \oplus (b + T)Y \subset K/T,$$

where $\mathbb{Q}^{(L)} \subset Y \subset \mathbb{Q}$ and the type $\mathbf{t}(Y)$ is nonidempotent.

The ring $R \subset K$ defined by $R/T = \overline{R}$ is a mixed E-ring of rank 2 whose additive group is not q.d.

E-ring of rank 2 whose additive group is not q.d.

$\mathbb{Q}^{(L)}$ is the subring of $\mathbb{Q}$ generated by all $\frac{1}{p}$, where $p \in L$.

Example 24. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \qquad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K,$$

$b = (p + p^2\mathbb{Z})_{p \in L} \in K$. Denote

$$\overline{R} = (1+T)\mathbb{Q}^{(L)} \oplus (b+T)Y \subset K/T,$$

where $\mathbb{Q}^{(L)} \subset Y \subset \mathbb{Q}$ and the type $\mathbf{t}(Y)$ is nonidempotent.

The ring $R \subset K$ defined by $R/T = \overline{R}$ is a mixed E-ring of rank 2 whose additive group is not q.d.

Remark. $\overline{R} = R/T$ is not an E-ring.

Definition. A group G with $T(G)$ reduced is *generalized quotient divisible* if it is an extension of a free group by a divisible torsion group.

Definition. A group G with $T(G)$ reduced is *generalized quotient divisible* if it is an extension of a free group by a divisible torsion group.

Proposition 25. *For a generalized E-ring R of rank $\geqslant 1$, the following are equivalent*:

1) *The additive group of R is a generalized q.d. group.*

2) *The additive group of $R/T(R)$ is a generalized q.d. group.*

Definition. A group G with $T(G)$ reduced is *generalized quotient divisible* if it is an extension of a free group by a divisible torsion group.

Proposition 25. *For a generalized E-ring R of rank $\geqslant 1$, the following are equivalent:*

1) *The additive group of R is a generalized q.d. group.*

2) *The additive group of $R/T(R)$ is a generalized q.d. group.*

By the result of J. D. Reid [1962], we obtain the following:

Corollary 26. *The additive group of any generalized E-ring of infinite rank is a generalized q.d. group.*

Definition. A group G with $T(G)$ reduced is *generalized quotient divisible* if it is an extension of a free group by a divisible torsion group.

Proposition 25. *For a generalized E-ring R of rank $\geqslant 1$, the following are equivalent:*

1) *The additive group of R is a generalized q.d. group.*

2) *The additive group of $R/T(R)$ is a generalized q.d. group.*

By the result of J. D. Reid [1962], we obtain the following:

Corollary 26. *The additive group of any generalized E-ring of infinite rank is a generalized q.d. group.*

Remark. Corollary 26 can be also deduced from the result of Tsarev [2021].

For a commutative ring R, the ideal of all nilpotent elements of R is denoted by $N(R)$.

For a commutative ring R, the ideal of all nilpotent elements of R is denoted by $N(R)$.

Theorem 27. *For an* E*-ring* R *of finite rank* $\geqslant 1$, *the following are equivalent*:

1) *The additive group of* R *is a q.d. group.*
2) *The additive group of* $R/T(R)$ *is a q.d. group.*
3) *The additive group of* $N\big(R/T(R)\big)$ *is a q.d. group.*

For a commutative ring R, the ideal of all nilpotent elements of R is denoted by $N(R)$.

Theorem 27. *For an E-ring R of finite rank $\geqslant 1$, the following are equivalent:*

1) *The additive group of R is a q.d. group.*

2) *The additive group of $R/T(R)$ is a q.d. group.*

3) *The additive group of $N\big(R/T(R)\big)$ is a q.d. group.*

Theorem 28. *For a torsion-free group G of finite rank, the following are equivalent:*

1) *There is a finite-rank E-ring R such that G is isomorphic to the additive group of $N\big(R/T(R)\big)$.*

2) *The set $\big\{p \in P \;\big|\; pG = G\big\}$ is infinite.*

For a commutative ring R, the ideal of all nilpotent elements of R is denoted by $N(R)$.

Theorem 27. *For an E-ring R of finite rank $\geqslant 1$, the following are equivalent:*

1) *The additive group of R is a q.d. group.*
2) *The additive group of $R/T(R)$ is a q.d. group.*
3) *The additive group of $N\big(R/T(R)\big)$ is a q.d. group.*

Theorem 28. *For a torsion-free group G of finite rank, the following are equivalent:*

1) *There is a finite-rank E-ring R such that G is isomorphic to the additive group of $N\big(R/T(R)\big)$.*
2) *The set $\big\{p \in P \;\big|\; pG = G\big\}$ is infinite.*

It follows from Theorems 27 and 28 that there is a sufficient supply of E-rings whose additive groups are not q.d.

p-components of E-rings

Let R be an E-ring and R_p be its p-components.

p-components of E-rings

Let R be an E-ring and R_p be its p-components.

Theorem 29 [Sch, 1973]. *All R_p's are cyclic groups.*

p-components of E-rings

Let R be an E-ring and R_p be its p-components.

Theorem 29 [Sch, 1973]. *All R_p's are cyclic groups.*

Theorem 30.

If R has finite rank and its additive group is not q.d., then

a) *$N(R)$ is a mixed group with infinite torsion part;*

p-components of E-rings

Let R be an E-ring and R_p be its p-components.

Theorem 29 [Sch, 1973]. *All R_p's are cyclic groups.*

Theorem 30.
If R has finite rank and its additive group is not q.d., then
a) *$N(R)$ is a mixed group with infinite torsion part;*
b) *the set $\left\{p \in P \;\middle|\; p^2 \leqslant |R_p|\right\}$ is infinite.*

p-components of E-rings

Let R be an E-ring and R_p be its p-components.

Theorem 29 [Sch, 1973]. *All R_p's are cyclic groups.*

Theorem 30.

If R has finite rank and its additive group is not q.d., then

a) *$N(R)$ is a mixed group with infinite torsion part;*

b) *the set $\left\{p \in P \,\middle|\, p^2 \leqslant |R_p|\right\}$ is infinite.*

Theorem 31 [Sch, 1973]. Let $p \in P$.

a) *There is a unique R'_p such that $R = R_p \oplus R'_p$.*

b) *R'_p is an ideal of R and an E-ring.*

c) *If $R_p \neq 0$, then $pR'_p = R'_p$.*

E-ring as an extension of an L-divisible ideal

$\boldsymbol{E}$-ring as an extension of an $\boldsymbol{L}$-divisible ideal

For any $r \in R$ and $p \in L$, where $L = \left\{p \in P \;\middle|\; R_p \neq 0\right\}$, we can write $r = r_p + r'_p$ with $r_p \in R_p$ and $r'_p \in R'_p$.

$\boldsymbol{E}$-ring as an extension of an $\boldsymbol{L}$-divisible ideal

For any $r \in R$ and $p \in L$, where $L = \{p \in P \mid R_p \neq 0\}$, we can write $r = r_p + r'_p$ with $r_p \in R_p$ and $r'_p \in R'_p$.

Define $\xi\colon R \to \prod_{p \in L} R_p$ by putting $\xi(r) = (r_p)_{p \in L}$.

$\boldsymbol{E}$-ring as an extension of an $\boldsymbol{L}$-divisible ideal

For any $r \in R$ and $p \in L$, where $L = \{p \in P \mid R_p \neq 0\}$, we can write $r = r_p + r'_p$ with $r_p \in R_p$ and $r'_p \in R'_p$.

Define $\xi\colon R \to \prod_{p \in L} R_p$ by putting $\xi(r) = (r_p)_{p \in L}$.

Theorem 32 [Sch, 1973].

a) $\xi(R)$ *is an E-ring such that* $\bigoplus_{p \in L} R_p \subset \xi(R) \subset \prod_{p \in L} R_p$.

b) $\ker \xi$ *is the (torsion-free) ideal* $A = \bigcap_{p \in L} \bigcap_{n \in \mathbb{N}} p^n R$.

E-ring as an extension of an *L*-divisible ideal

For any $r \in R$ and $p \in L$, where $L = \{p \in P \mid R_p \neq 0\}$, we can write $r = r_p + r'_p$ with $r_p \in R_p$ and $r'_p \in R'_p$.

Define $\xi \colon R \to \prod_{p \in L} R_p$ by putting $\xi(r) = (r_p)_{p \in L}$.

Theorem 32 [Sch, 1973].

a) $\xi(R)$ *is an E-ring such that* $\bigoplus_{p \in L} R_p \subset \xi(R) \subset \prod_{p \in L} R_p$.

b) $\ker \xi$ *is the (torsion-free) ideal* $A = \bigcap_{p \in L} \bigcap_{n \in \mathbb{N}} p^n R$.

Theorem 33 [Ts, 2017].

If R has a finite rank, then $\big(N(R) \cap A\big)^2 = 0$.

E-ring as an extension of an L-divisible ideal

For any $r \in R$ and $p \in L$, where $L = \{p \in P \mid R_p \neq 0\}$, we can write $r = r_p + r'_p$ with $r_p \in R_p$ and $r'_p \in R'_p$.

Define $\xi \colon R \to \prod_{p \in L} R_p$ by putting $\xi(r) = (r_p)_{p \in L}$.

Theorem 32 [Sch, 1973].

a) $\xi(R)$ *is an* E*-ring such that* $\bigoplus_{p \in L} R_p \subset \xi(R) \subset \prod_{p \in L} R_p$.

b) $\ker \xi$ *is the (torsion-free) ideal* $A = \bigcap_{p \in L} \bigcap_{n \in \mathbb{N}} p^n R$.

Theorem 33 [Ts, 2017].
If R *has a finite rank, then* $\big(N(R) \cap A\big)^2 = 0$.

Theorem 34. *If* R *has a finite rank, then* $N(R) \cdot A = 0$.

Example 35 [BSch, 1977].

Example 35 [BSch, 1977].

Let $L = P \setminus \{2, 3\}$ and $\chi = (\infty_2, 0_3, 1_5, 1_7, \ldots)$; then

$$\bigoplus_{p>3} \mathbb{Z}/p\mathbb{Z} \subset R^\chi \subset \prod_{p>3} \mathbb{Z}/p\mathbb{Z}$$

and $|R^\chi/2R^\chi| = 2 = |\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)}|$.

Example 35 [BSch, 1977].

Let $L = P \setminus \{2,3\}$ and $\chi = (\infty_2, 0_3, 1_5, 1_7, \ldots)$; then

$$\bigoplus_{p>3} \mathbb{Z}/p\mathbb{Z} \subset R^\chi \subset \prod_{p>3} \mathbb{Z}/p\mathbb{Z}$$

and $|R^\chi/2R^\chi| = 2 = |\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)}|$.

If we put $R = \left\{(a,b) \in \mathbb{Q}^{(L)} \oplus R^\chi \mid \varphi(a + \mathbb{Q}^{(L)}) = b + R^\chi\right\}$, where $\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)} \overset{\varphi}{\cong} R^\chi/2R^\chi$, then R is an E-ring of rank 2.

Example 35 [BSch, 1977].

Let $L = P \setminus \{2,3\}$ and $\chi = (\infty_2, 0_3, 1_5, 1_7, \dots)$; then

$$\bigoplus_{p>3} \mathbb{Z}/p\mathbb{Z} \subset R^{\chi} \subset \prod_{p>3} \mathbb{Z}/p\mathbb{Z}$$

and $|R^{\chi}/2R^{\chi}| = 2 = |\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)}|$.

If we put $R = \left\{(a,b) \in \mathbb{Q}^{(L)} \oplus R^{\chi} \;\middle|\; \varphi(a + \mathbb{Q}^{(L)}) = b + R^{\chi}\right\}$, where $\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)} \overset{\varphi}{\cong} R^{\chi}/2R^{\chi}$, then R is an E-ring of rank 2.

For this ring we have $\xi(R) \cong R^{\chi}$ and $A = \ker \xi = 2\mathbb{Q}^{(L)} \oplus 0$.

Example 35 [BSch, 1977].

Let $L = P \setminus \{2,3\}$ and $\chi = (\infty_2, 0_3, 1_5, 1_7, \ldots)$; then

$$\bigoplus_{p>3} \mathbb{Z}/p\mathbb{Z} \subset R^\chi \subset \prod_{p>3} \mathbb{Z}/p\mathbb{Z}$$

and $|R^\chi/2R^\chi| = 2 = |\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)}|$.

If we put $R = \left\{(a,b) \in \mathbb{Q}^{(L)} \oplus R^\chi \;\middle|\; \varphi(a + \mathbb{Q}^{(L)}) = b + R^\chi\right\}$, where $\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)} \overset{\varphi}{\cong} R^\chi/2R^\chi$, then R is an E-ring of rank 2.

For this ring we have $\xi(R) \cong R^\chi$ and $A = \ker \xi = 2\mathbb{Q}^{(L)} \oplus 0$.

Thus the exact sequence $0 \to A \to R \to \xi(R) \to 0$ does not split.

Example 35 [BSch, 1977].

Let $L = P \setminus \{2,3\}$ and $\chi = (\infty_2, 0_3, 1_5, 1_7, \ldots)$; then

$$\bigoplus_{p>3} \mathbb{Z}/p\mathbb{Z} \subset R^\chi \subset \prod_{p>3} \mathbb{Z}/p\mathbb{Z}$$

and $|R^\chi/2R^\chi| = 2 = |\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)}|$.

If we put $R = \left\{(a,b) \in \mathbb{Q}^{(L)} \oplus R^\chi \;\middle|\; \varphi(a + \mathbb{Q}^{(L)}) = b + R^\chi\right\}$, where $\mathbb{Q}^{(L)}/2\mathbb{Q}^{(L)} \overset{\varphi}{\cong} R^\chi/2R^\chi$, then R is an E-ring of rank 2.

For this ring we have $\xi(R) \cong R^\chi$ and $A = \ker \xi = 2\mathbb{Q}^{(L)} \oplus 0$.

Thus the exact sequence $0 \to A \to R \to \xi(R) \to 0$ does not split.

On the other hand, R is quasi-isomorphic to $\mathbb{Q}^{(L)} \oplus \xi(R)$.

Problem [BSch, 1977]. Is it true that every mixed E-ring R is quasi-isomorphic to a direct sum of $\xi(R)$ and a torsion-free E-ring containing the ideal $A = \ker \xi$?

Problem [BSch, 1977]. Is it true that every mixed E-ring R is quasi-isomorphic to a direct sum of $\xi(R)$ and a torsion-free E-ring containing the ideal $A = \ker \xi$?

It follows from the results of Tsarev [2017] that the answer is positive if R has rank $\leqslant 2$.

Problem [BSch, 1977]. Is it true that every mixed E-ring R is quasi-isomorphic to a direct sum of $\xi(R)$ and a torsion-free E-ring containing the ideal $A = \ker \xi$?

It follows from the results of Tsarev [2017] that the answer is positive if R has rank $\leqslant 2$.

We construct an E-ring R of rank 3 with the following properties:

- $R \subset \mathbb{Q} \times \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}$;
- R is a counterexample to the conjecture of Bowshell and Schultz;

Problem [BSch, 1977]. Is it true that every mixed E-ring R is quasi-isomorphic to a direct sum of $\xi(R)$ and a torsion-free E-ring containing the ideal $A = \ker \xi$?

It follows from the results of Tsarev [2017] that the answer is positive if R has rank $\leqslant 2$.

We construct an E-ring R of rank 3 with the following properties:

- $R \subset \mathbb{Q} \times \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}$;
- R is a counterexample to the conjecture of Bowshell and Schultz;
- the additive group of R is not q.d.;
- the additive group of the ring $\xi(R) \cong R/A$ is q.d.

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Choose $x, y \in K$ such that $x^2 = y^2 = xy = 0$ and 1, x, y are independent.

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Choose $x, y \in K$ such that $x^2 = y^2 = xy = 0$ and 1, x, y are independent.
We consider the ring $\underbrace{\overline{1}\,\mathbb{Q}^{(L)} \oplus \overline{x}\,\mathbb{Q}}_{\overline{\boldsymbol{U}}} \oplus \underbrace{\overline{y}\,\mathbb{Q}}_{\overline{\boldsymbol{I}}} \subset K/T.$

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Choose $x, y \in K$ such that $x^2 = y^2 = xy = 0$ and 1, x, y are independent.
We consider the ring $\underbrace{\overline{1}\,\mathbb{Q}^{(L)} \oplus \overline{x}\,\mathbb{Q}}_{\overline{\boldsymbol{U}}} \oplus \underbrace{\overline{y}\,\mathbb{Q}}_{\overline{\boldsymbol{I}}} \subset K/T$.

Define $U \subset K$ by $U/T = \overline{U}$. For a group $H \subset \mathbb{Q} \oplus \mathbb{Q}$, let

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Choose $x, y \in K$ such that $x^2 = y^2 = xy = 0$ and 1, x, y are independent.
We consider the ring $\underbrace{\overline{1}\,\mathbb{Q}^{(L)} \oplus \overline{x}\,\mathbb{Q}}_{\overline{\boldsymbol{U}}} \oplus \underbrace{\overline{y}\,\mathbb{Q}}_{\overline{\boldsymbol{I}}} \subset K/T$.

Define $U \subset K$ by $U/T = \overline{U}$. For a group $H \subset \mathbb{Q} \oplus \mathbb{Q}$, let

$$\overline{\Lambda} = \left\{ \overline{1} \cdot q + \overline{x} \cdot a + \overline{y} \cdot b \;\middle|\; q \in \mathbb{Q}^{(L)} \text{ and } (a, b) \in H \right\} \subset \overline{U} \oplus \overline{I}.$$

Example 36. Let $L \subset P$ and $P \setminus L$ be infinite and

$$K = \prod_{p \in L} \mathbb{Z}/p^2\mathbb{Z}, \quad T = \bigoplus_{p \in L} \mathbb{Z}/p^2\mathbb{Z} \subset K, \quad \overline{k} = k + T.$$

Choose $x, y \in K$ such that $x^2 = y^2 = xy = 0$ and 1, x, y are independent.
We consider the ring $\underbrace{\overline{1}\,\mathbb{Q}^{(L)} \oplus \overline{x}\,\mathbb{Q}}_{\overline{\boldsymbol{U}}} \oplus \underbrace{\overline{y}\,\mathbb{Q}}_{\overline{\boldsymbol{I}}} \subset K/T$.

Define $U \subset K$ by $U/T = \overline{U}$. For a group $H \subset \mathbb{Q} \oplus \mathbb{Q}$, let

$$\overline{\Lambda} = \left\{ \overline{1} \cdot q + \overline{x} \cdot a + \overline{y} \cdot b \;\middle|\; q \in \mathbb{Q}^{(L)} \text{ and } (a,b) \in H \right\} \subset \overline{U} \oplus \overline{I}.$$

Then $R = \left\{ \begin{pmatrix} u & z \\ 0 & u \end{pmatrix} \;\middle|\; u \in U,\ z \in \overline{I} \text{ and } \overline{u} + z \in \overline{\Lambda} \right\}$ is the desired E-ring with $A \subset \begin{pmatrix} 0 & \overline{I} \\ 0 & 0 \end{pmatrix}$ (for a suitable H).

[BW-1959] R. A. Beaumont, R. J. Wisner. *Rings with additive group which is a torsion-free group of rank two* // Acta Sci. Math. Szeged, 1959, **20**:2-3, 105–116.

[BP-1961a] R. A. Beaumont, R. S. Pierce. *Torsion-free rings* // Illinois J. Math., 1961, **5**:1, 61–98.

[BP-1961b] R. A. Beaumont, R. S. Pierce. *Torsion-free groups of rank two* // Memoirs AMS, 1961, **38**.

[BS-1977] R. A. Bowshell, P. Schultz. *Unital rings whose additive endomorphisms commute* // Math. Ann., 1977, **228**:3, 197–214.

[D-2007] О. И. Давыдова. *Факторно делимые группы ранга* 1 // Фундамент. и прикл. мат., 2007, **13**:3, 25–33.

[F-1984] A. A. Fomin. *Abelian groups with free subgroups of infinite index and their endomorphism rings* // Math. Notes, 1984, **36**:2, 581–585.

[FW-1998] A. A. Fomin, W. Wickless. *Quotient divisible Abelian groups* // Proc. Amer. Math. Soc., 1998, **126**:1, 45–52.

[F-2009] A. A. Fomin. *Invariants for Abelian groups and dual exact sequences* // J. Algebra, 2009, **322**:7, 2544–2565.

[Fuchs-1952] L. Fuchs. *On subdirect unions, I* // Acta Math. Acad. Sci. Hung., 1952, **3**:1-2, 103–120.

[Fuchs-2015] L. Fuchs. *Abelian Groups*, Springer, 2015.

[GSS-2004] R. Göbel, S. Shelah, L. Strüngmann. *Generalized E-rings // Rings, Modules, Algebras, and Abelian Groups*, Marcel Dekker, 2004, 291–306.

[R-1962] J. D. Reid. *A note on torsion-free Abelian groups of infinite rank* // Proc. AMS, 1962, **13**:2, 222–225.

[S-1973] P. Schultz. *The endomorphism ring of the additive group of a ring* // J. Austral. Math. Soc., 1973, **15**:1, 60–69.

[Ts-2017] А. В. Царев. *E-кольца малых рангов* // Чебышевский сб., 2017, **18**:2, 235–244.

[Ts-2021] A. V. Tsarev. *A generalization of quotient divisible groups to the infinite rank case* // Sib. Math. J., 2021, **62**:3, 554–559.

[ZT-2021] M. N. Zonov, E. A. Timoshenko. *Quotient divisible groups of rank* 2 // Math. Notes, 2021, **110**:1, 48–60.