Integrable systems and neural networks

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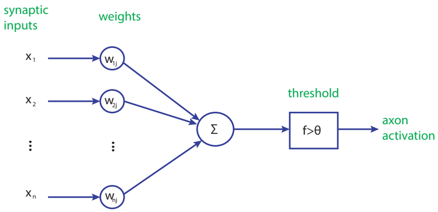
1. **Definitions**
   - Hopfield model
   - Zamolodchikov equation
   - Ising model
2. **Vertex representation**
   - 3D Ising model
   - Hopfield model on triangular lattice
   - TTE and ∞-simplex equation
3. **Mutations in weighted graphs**
   - Electrical networks
   - Star-triangle transformation
   - Isolated Star-triangle transformation

Let Γ be a graph with N vertices (neurons);
a connectivity matrix w_{ij} , a function $w : E \to \mathbb{R}$; a state $x : V \to \{\pm 1\}$.
The state transition probability is given by

$$P(x', x) = \prod_i (1 + e^{-\beta x_i' (\sum_j w_{ij} x_j - t_i)})^{-1}.$$



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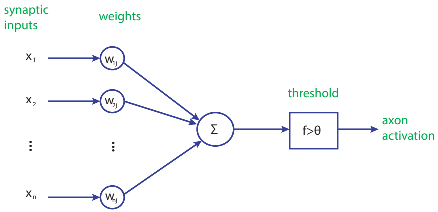

The learning stage on the set of m patterns $\{\epsilon^1, \dots, \epsilon^m\}$ with $\epsilon^k = (\epsilon^k_1, \dots, \epsilon^k_n)$ is provided by the following weights

$$w_{ij} = \frac{1}{n} \sum_{k=1}^{m} \epsilon^k_i \epsilon^k_j .$$

Yang-Baxter equation

$R \in End(V^{\otimes 2})$, where V - (f.d) vector space. T

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in End(V^{\otimes 3})$$

R_{ij} represents the operator acting in components i, j as R and trivially in the other.

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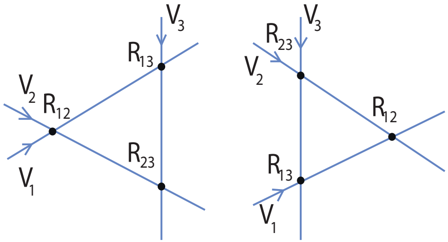

TE

$\Phi \in End(V^{\otimes 3})$, where V - (f.d) vector space.

$$\Phi_{123}\Phi_{145}\Phi_{246}\Phi_{356} = \Phi_{356}\Phi_{246}\Phi_{145}\Phi_{123} \in End(V^{\otimes 6})$$

Φ_{ijk} represents the operator acting in components i, j, k as Φ and trivially in the others.

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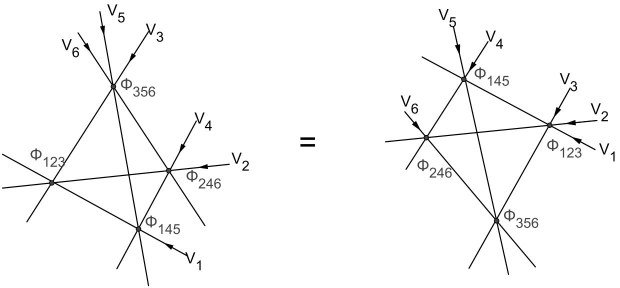

Figure: Tetrahedral equation

Electric solution

$$\Phi(x,y,z) = (x_1, y_1, z_1);$$
$$x_1 = \frac{xy}{x+z+xyz},$$
$$y_1 = x+z+xyz,$$
$$z_1 = \frac{yz}{x+z+xyz},$$

is a rational transformation, it acts on the space of rational functions on variables x, y, z.

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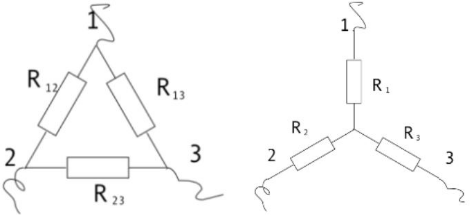

Figure: Star-triangle transformation

n-simplex equation

Definition

The set theoretic n-simplex equation on the set X is the following equality for the composition of R-maps (or correspondences) acting from right to left

$$\cdots \circ R_{(**0*\ldots*)} \circ R_{(*1**\ldots*)} \circ R_{(0***\ldots*)} = R_{(1***\ldots*)} \circ R_{(*0**\ldots*)} \circ R_{(**1*\ldots*)} \circ \cdots .$$

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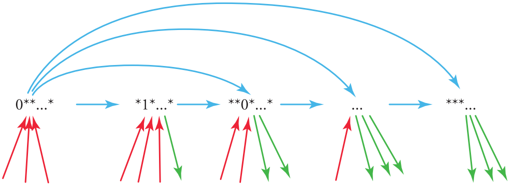

Figure: I-configuration

Γ, $a : E \to \mathbb{R}$ is an interaction parameter.
$\sigma : V \to \{\pm 1\}$

$$Z = \sum_{\sigma} \prod_{e \in E} (1 + a_e \delta(\sigma_e^1, \sigma_e^2))$$

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$$Z = \sum_{\sigma} \prod_{e \in E} (1 + a_e \delta(\sigma_e^1, \sigma_e^2))$$

The probability of a state σ is given by

$$P(\sigma) = \frac{1}{Z} \prod_{e \in E} (1 + a_e \delta(\sigma_e^1, \sigma_e^2))$$

The 2-point correlation matrix is given by

$$< \sigma_i, \sigma_j >= \sum_{\sigma} P(\sigma) \sigma_i \sigma_j$$

Time evolution

Consider the Hopfield model on a triangular lattice

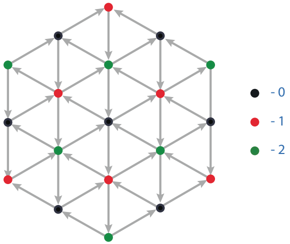

Figure: Triangular lattice

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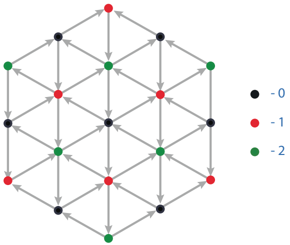

Figure: Triangular lattice

This lattice can be considered as a projection of the cubic lattice on the plane $i + j + k = 0$.

The conditional probability that the model will pass through a set of states with free initial data is:

$$
\begin{aligned}
P &= \prod_{i+j+k=a}^{b} \left(1 + \exp(x_{ijk}(w_{ijk}^1 x_{i-1jk} + w_{ijk}^2 x_{ij-1k} + w_{ijk}^3 x_{ijk-1})\right)^{-1} \\
&= \prod_{i+j+k=a}^{b} \frac{\exp((x_{ijk}(w_{ijk}^1 x_{i-1jk} + w_{ijk}^2 x_{ij-1k} + w_{ijk}^3 x_{ijk-1})/2)}{2\cosh((x_{ijk}(w_{ijk}^1 x_{i-1jk} + w_{ijk}^2 x_{ij-1k} + w_{ijk}^3 x_{ijk-1})/2)}.
\end{aligned}
$$

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Lemma

Let us define the map $f(w^1,w^2,w^3)=(w^0,w^{12},w^{13},w^{23})$ *by the following equation*

$$(\cosh((w^1s_1+w^2s_2+w^3s_3)/2))^{-1} = \exp\left((w^0+w^{12}s_1s_2+w^{13}s_1s_3+w^{23}s_2s_3)/2\right)$$

$\forall s_i=\pm 1.$

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Lemma

Let us define the map $f(w^1, w^2, w^3) = (w^0, w^{12}, w^{13}, w^{23})$ by the following equation

$$(\cosh((w^1 s_1 + w^2 s_2 + w^3 s_3)/2))^{-1} = \exp\left((w^0 + w^{12} s_1 s_2 + w^{13} s_1 s_3 + w^{23} s_2 s_3)/2\right)$$

$\forall s_i = \pm 1.$

Remark

The transformation $F : (w^1, w^2, w^3) \mapsto (w^{12}, w^{13}, w^{23})$ is known in the theory of the Ising model as the star-triangle transformation. This transformation is a solution to the Zamolodchikov tetrahedron equation.

The conditional probability 1 coincides with the partition function of the Ising model on a regular cubic lattice with additional diagonal edges, shown in the figure 5:

$$
\begin{aligned}
P &= \prod_{i+j+k=a}^{b} \exp\left((x_{ijk}(w^1_{ijk}x_{i-1jk} + w^2_{ijk}x_{ij-1k} + w^3_{ijk}x_{ijk-1})/2\right) \times \\
&\times \prod_{i+j+k=a}^{b} \exp\left((w^{12}_{ijk}x_{i-1jk}x_{ij-1k} + w^{13}_{ijk}x_{i-1jk}x_{ijk-1} + w^{23}_{ijk}x_{ij-1k}x_{ijk-1})/2\right)
\end{aligned}
$$

where $(w^{12}_{ijk}, w^{13}_{ijk}, w^{23}_{ijk}) = F(w^1_{ijk}, w^2_{ijk}, w^3_{ijk})$.

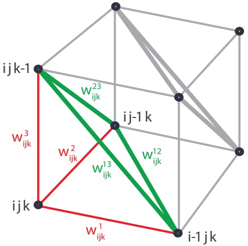

Figure: Cubic lattice

First we pass to the variables s_{ij} on edges and associate the space $\mathbb{C}^2$ to each edge

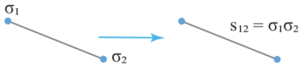

Figure: Edge variables

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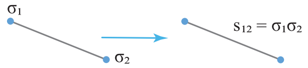

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with a condition $s_1 s_2 s_3 s_4 = 1$ on all 2-faces.
Then we consider the dual lattice Λ^*

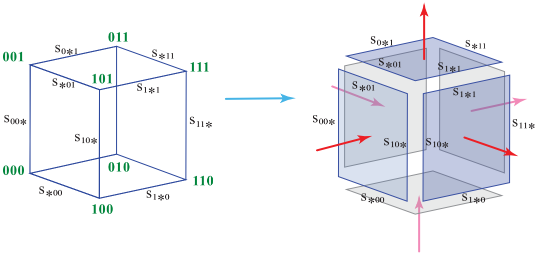

Figure: Dual graph

Lemma

The associated matrix W_0 satisfies the matrix tetrahedral equation.

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One could find a particular set of edges of each 3-cube of the configuration

Table: Left side of the TE

0***	*1**	**0*	***1
01*0 000* 0*11	011* *100 11*1	0*00 110* *001	00*1 *111 1*01

Table: Right side of the TE

***0	**1*	*0**	1***
1*00 00*0 *110	0*10 111* *011	001* *000 10*1	100* 1*11 11*0

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0***	*1**	**0*	***1
01*0	011*	0*00	00*1
000*	*100	110*	*111
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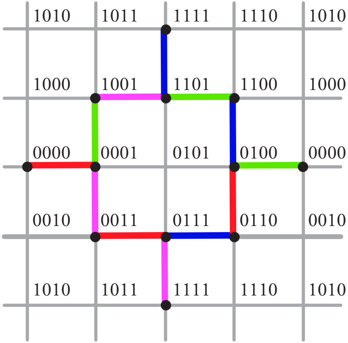

Figure: 12 edges in 4 3-cubes

Definition

The weight function W is defined as a matrix with matrix elements indexed by 3 2-faces of a 3-cube

$$W = W_0 \times \exp(t \sum_{u=1}^{3} s_u)$$

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Lemma

The matrix W provides a weight matrix for the 3D Ising model. This means that the partition function $Z(t)$ can be obtained as a product

$$Z(t) = \prod_{(\alpha,\beta,\gamma)\in\Lambda^*} W_{\alpha\beta\gamma}$$

over the dual lattice Λ^*.

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Theorem

The weight matrix W satisfies an equation

$$\begin{aligned} &P_{16} P_{25} W^A_{123} W^A_{145} W^A_{246} W^A_{356} P_{16} P_{25} \\ &= W_{356} W_{246} W_{145} W_{123}. \end{aligned}$$

Let us attach a spin variable to each diagonal by the same rule as for the edge-spins - by the product of the vertex-spins in its ends.

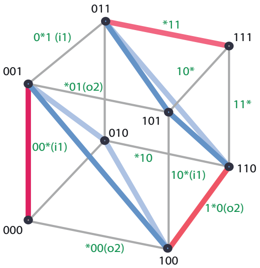

Figure: Diagonal choice

Lemma

For each 2-face of a 3-cube the value of the diagonal spin is related with the edge-spins by the formula

$$s = s_{i_1} \times s_{o_2} = s_{i_2} \times s_{o_1}.$$

The choice of the diagonals on the left and right hand sides of the tetrahedron equation are given by the figures.

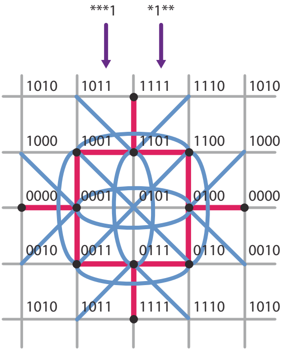

Figure: Diagonals on LHS

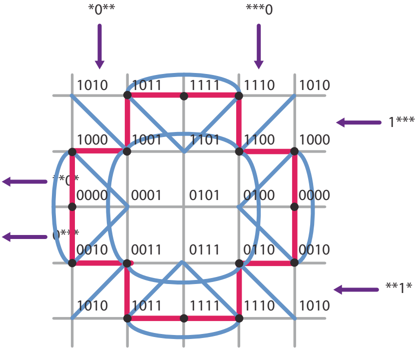

Figure: Diagonals on RHS

Definition

Let us define the weight matrix W_h which is constructed with a help of the 3D Ising model weight matrix W as follows

$$W_h = W \times \exp(\gamma \sum_f \sigma_{i_1} \sigma_{o_2}).$$

Here the sum is over 2-faces of a 3-cube, and the spins variables correspond to the edges of a 2-face.

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$$= W_{356} W_{246} W_{145} W_{123}.$$

By an analogy with the A_∞ algebras we define an ∞-simplex structure (or homotopy-simplex structure) as a set of multi-operations:

$$\Phi^{(k)}: V^{\otimes k} \to V^{\otimes k}$$

such that

$$\Phi^{(k)}_{1\ldots k}\left(\Phi^{(k-1)}_{1\ldots k-1}\Phi^{(k-1)}_{1\ldots k-2,k}\ldots\Phi^{(k-1)}_{2\ldots k}\right) = \left(\Phi^{(k-1)}_{2\ldots k}\ldots\Phi^{(k-1)}_{1\ldots k-2,k}\Phi^{(k-1)}_{1\ldots k-1}\right)\Phi^{(k)}_{1\ldots k}$$

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Theorem

A solution W for the TTE provides an ∞-simplex structure with the following maps:

$$\begin{aligned}
\Phi^{(3)} &= W : V^{\otimes 3} \to V^{\otimes 3},\\
\Phi^{(4)} &= Ad_A \circ P_{14}P_{23} : V^{\otimes 4} \to V^{\otimes 4},\\
\Phi^{(5)} &= P_{35} : V^{\otimes 5} \to V^{\otimes 5},\\
\Phi^{(k)} &= id \qquad \forall k > 5.
\end{aligned}$$

Γ - connected graph without loops
$V_B \subset V = \{1, \dots, n\}$.
$a : E \to \mathbb{R}^+$

Definition

The Kirchhoff matrix is

$$T_{ij} = \begin{cases} -a_{ij} & \text{if} \quad i \neq j \\ \sum_{k \neq i} a_{ik} & \text{if} \quad i = j \end{cases}$$

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A problem: to find currents in boundary vertices $I : V_B \to \mathbb{R}$ for given potentials $U : V_B \to \mathbb{R}$
Let us present the Kirchhoff matrix in a block form:

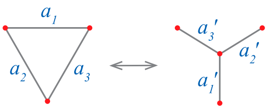
$$T = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

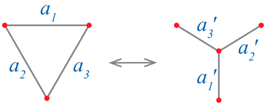
$$I = M_R U,$$

with the response matrix

$$M_R = A - BC^{-1}B^T,$$

which is the Schur complement of C.


a_1
a_2
a_3
a_3'
a_2'
a_1'



Electrical networks:

$$a_i' a_i = a_1 a_2 + a_1 a_3 + a_2 a_3$$

Ising model:

$$\begin{aligned} a_1' a_2' &= \frac{1 + a_1 a_2 a_3}{a_1 a_2 + a_3} \\ a_1' a_3' &= \frac{1 + a_1 a_2 a_3}{a_1 a_3 + a_2} \\ a_2' a_3' &= \frac{1 + a_1 a_2 a_3}{a_2 a_3 + a_1} \end{aligned}$$

Figure: Star-triangle transformation

where $s_i = \exp(w_{i4}),\ r_1 = \exp(w_{23}),\ r_2 = \exp(w_{13}),\ r_3 = \exp(w_{12})$ and the following system of equations fulfills

$$
\begin{aligned}
\frac{s_1 + s_2 s_3}{1 + s_1 s_2 s_3} &= \left(\frac{r_1 + r_2}{1 + r_1 r_2}\right)\left(\frac{r_1 + r_3}{1 + r_1 r_3}\right) \\
\frac{s_2 + s_1 s_3}{1 + s_1 s_2 s_3} &= \left(\frac{r_1 + r_2}{1 + r_1 r_2}\right)\left(\frac{r_2 + r_3}{1 + r_2 r_3}\right) \\
\frac{s_3 + s_1 s_2}{1 + s_1 s_2 s_3} &= \left(\frac{r_1 + r_3}{1 + r_1 r_3}\right)\left(\frac{r_2 + r_3}{1 + r_2 r_3}\right)
\end{aligned}
$$

Parallel-serial connection

Figure: Parallel connection

Parallel-serial connection

Figure: Parallel connection

Figure: Serial connection

Thank you!