n-simplex equations and corresponding algebraic systems

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Contents

§ 1. Yang–Baxter equation and bi-groupoids

Let X be a set. A map

$$R : X \times X \to X \times X$$

is said to be a set-theoretic solution or simply solution for the Yang–Baxter equation (YBE):

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} : X^3 \to X^3$ acts as R on the i-th and j-th factors and as identity map on the other factor.

Example

For arbitrary set X the map $P(x, y) = (y, x)$ gives a solution for the Yang–Baxter equation.

If a map

$$R : X \times X \to X \times X$$

gives a solution for YBE, then the operator

$$S = PR : X \times X \to X \times X$$

gives a solution for the Braid equation,

$$S_1 S_2 S_1 = S_2 S_1 S_2,$$

where $S_1 = S \times id$ and $S_2 = id \times S$ are operators on $X \times X \times X$.

This equation corresponds to relation in the braid group and the third Reidemeister move.

Writing

$$R(x, y) = (\sigma_y(x), \tau_x(y))$$

for $x, y \in X$, we say that a solution (X, R) is

– non-degenerate if σ_x and τ_x are invertible for all $x \in X$;

– square-free if $R(x, x) = (x, x)$ for all $x \in X$;

– involutive if $R^2 = id$.

A groupoid is a non-empty set with one binary algebraic operation.

A bi-groupoid is a non-empty set with two binary algebraic operations.

If (X, R), $R(x, y) = (\sigma_y(x), \tau_x(y))$, is a solution of the YBE, then we can define a bi-groupoid

$$(X; \cdot, *),$$

where

$$\cdot, * : X \times X \to X, \quad x \cdot y = \sigma_x(y), \quad y * x = \tau_y(x).$$

Proposition

Let $(X, \cdot, *)$ be a bi-groupoid and $R : X \times X \to X \times X$ given by $R(x, y) = (x \cdot y, y * x)$ for $x, y \in X$. Then the pair (X, R) is a solution of YBE if and only if the equalities

$$
\begin{aligned}
(x \cdot y) \cdot z &= (x \cdot (z * y)) \cdot (y \cdot z), \\
(y * x) \cdot (z * (x \cdot y)) &= (y \cdot z) * (x \cdot (z * y)), \\
(z * (x \cdot y)) * (y * x) &= (z * y) * x,
\end{aligned}
$$

hold for all $x, y, z \in X$.

Corollary

1. If $x \cdot y = y$ for all $x, y \in X$, then the pair (X, R) is a solution of the YBE if and only if the operation $*$ is right distributive, i.e.

$$(z * x) * (y * x) = (z * y) * x$$

for all $x, y, z \in X$.

2. If $y * x = y$ for all $x, y \in X$, then the pair (X, R) is a solution of the YBE if and only if the operation $\cdot$ is right distributive, i.e.

$$(z \cdot y) \cdot x = (z \cdot x) \cdot (y \cdot x)$$

for all $x, y, z \in X$.

If $\sigma_y = id$ for all $y \in X$ or $\tau_x = id$ for all $x \in X$, then the solution (X, R) is called by elementary solution.

Any elementary solution defines a groupoid structure on X.

If $R(x, y) = (\sigma_y(x), y)$ and $P(x, y) = (y, x)$, then

$$P\,R\,P(x, y) = (x, \sigma_x(y)).$$

A. Soloviev (2000) proves that any non-degenerate solution is conjugate to an elementary solution.

Proposition (A. Soloviev, 2000)

If $R(x,y) = (\sigma_y(x), \tau_x(y))$, $x, y \in X$, gives a non-degenerate solution for YBE on X, then it conjugates to a solution of the form:

$$R'(x,y) = (\sigma_x(\tau_{\sigma_y^{-1}(x)}(y)),\ y).$$

If for all $a, b \in X$ there exists a unique $x \in X$ such that

$$\tau_{\sigma_x^{-1}(a)}(x) = \sigma_a^{-1}(b),$$

then this solution is non-degenerate.

– Any elementary solution of the YBE defines a right distributive groupoid;

– any non-degenerate solution defines a rack;

– any non-degenerate square-free solution define a quandle.

§ 2. Racks and quandles

A quandle is a groupoid which satisfies three axioms.

These axioms motivated by the three Reidemeister moves of diagrams of knots in the Euclidean space $\mathbb{R}^3$.

Quandles were introduced independently by S. Matveev and D. Joyce in 1982.

Definition

A rack is a non-empty set X with a binary algebraic operation

$$(a, b) \mapsto a * b$$

satisfying the following conditions:

(R1) For any $a, b \in X$ there is a unique $c \in X$ such that $a = c * b$;

(R2) Right distributivity: $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in X$.

A quandle X is a rack which satisfies the following condition:

(Q1) $a * a = a$ for all $a \in X$.

The simplest example of quandle is the so called trivial quandle.

A quandle X is called trivial if $a * b = a$ for all $a, b \in X$, i. e. any symmetry S_b is the trivial automorphism.

We see that a trivial quandle can contains arbitrary number of elements.

We shall denote the trivial quandle with n elements by T_n.

Many examples of quandles comes from groups.

Example

If G is a group and n is a natural number, then the set G equipped with the binary operations

$$a * b = b^{-n} a b^{n},$$

gives a quandle structure on G called the n-conjugation quandle, denoted by $Conj_n(G)$.

If G is abelian group, then $Conj_n(G)$ is a trivial quandle.

Example

If G is a group, then the set G equipped with the binary operations

$$a * b = ba^{-1}b,$$

gives a quandle structure on G called the core quandle, denoted by $Core(G)$.

We have seen that the n-conjugation quandle and the core quandle are defined by the words

$$u(a,b) = b^{-n}ab^n \text{ and } v(a,b) = ba^{-1}b,$$

respectively, in arbitrary group G.

We can formulate

Question

Let $w = w(x, y)$ be a a reduced word in the free group $F_2 = F_2(x, y)$. Under what conditions for arbitrary group G the algebraic system $(G, *_w)$ with binary operation

$$g *_w h = w(g, h)$$

is a rack (quandle)?

Theorem (V. B. – T. Nasybullov – M. Singh, 2019)

Let $w = w(x, y) \in F(x, y)$ be such that $Q = (G, *_w)$ is a rack for every group G. Then, in fact, Q is a quandle, and

$$w(x, y) = yx^{-1}y \text{ or } w(x, y) = y^{-n}xy^n \text{ for some } n \in \mathbb{Z}.$$

§ 3. n-simplex equations: construction and known solutions

Suppose that we have 3 straight lines l_1, l_2, and l_3 on the plane $\mathbb{R}^2$.

The line l_1 intersects with l_2 in the point R_{12}, with the line l_3 in the point R_{13}, and the line l_2 intersects with l_3 in the point R_{23}.

We assume that all points R_{12}, R_{13}, and R_{23} are different and are vertices of a triangle (2-simplex).

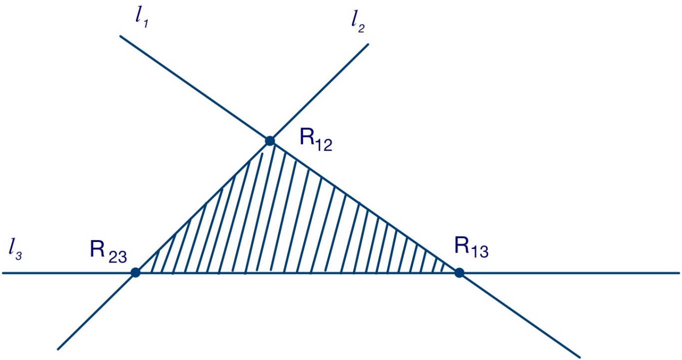

Figure: Geometric interpretation of YBE

Using the lexicographical order, we introduce the order on the vertices,

$$R_{12} < R_{13} < R_{23}.$$

Then the YBE is the equality of two words, where the first one is a word which we get if going around the vertices in the increasing order and the second word is a word which we get if going around the vertices in the decreasing order,

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}.$$

To get the tetrahedron equation (3-SE) we increment the indices of all lines by 3 and get the triangle with the vertices R_{45}, R_{46}, and R_{56}.

Further, embed our plane $\mathbb{R}^2$ into a 3-space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, take a vertex R_{123}, which does not lie in $\mathbb{R}^2$.

Construct a straight line l_1, which connect R_{123} with the first vertex R_{45}; construct a straight line l_2, which connect R_{123} with the second vertex R_{46} and construct a straight line l_3, which connect R_{123} with the third vertex, R_{56}.

We construct a tetrahedron with the vertices R_{123}, R_{145}, R_{246}, and R_{356}.

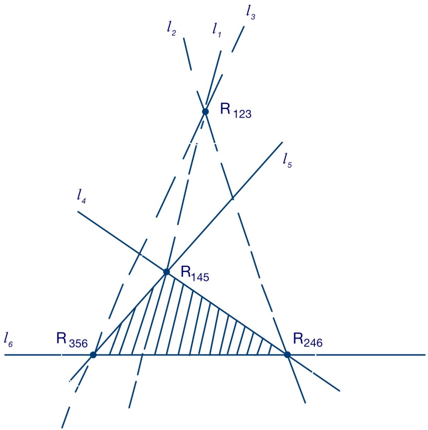

Figure: Geometric interpretation of TE

The TE or 3-SE is the equality of two words, where the first one is a word which we get if going around the vertices of the tetrahedron in the increasing order and the second word is a word which we get if going around the vertices in the decreasing order, i.e.

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}.$$

We have n-SE,

$$R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}} = R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}},$$

Define a shift

$$s_n : \mathbb{N} \to \mathbb{N}, \quad s_n(k) = k + (n+1),$$

and extend it to the multi-indexes by the rule, if

$$\overline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$$

is a multi-index, then

$$s_n(\overline{k}) = (s_n(k_1), s_n(k_2), \dots, s_n(k_n)) \in \mathbb{N}^n.$$

We get $(n+1)$-SE,

$$R_{1,2,\dots,n+1} R_{1,s_n(\overline{1})} R_{2,s_n(\overline{2})} \cdots R_{n+1,s_n(\overline{n})} =$$
$$= R_{n,s_n(\overline{n+1})} \cdots R_{2,s_n(\overline{2})} R_{1,s_n(\overline{1})} R_{1,2,\dots,n+1}.$$

Proposition

Let $R : X^n \to X^n$ be a solution of the n-SE.

1. If R is invertible, then its inverse R^{-1} is also a solution of the n-SE.
2. If $\sigma_1, \dots, \sigma_n$ are pairwise commuting endomorphisms of X, then a map $R : X^n \to X^n$ defined as

$$R(x_1, \dots, x_n) = (\sigma_1(x_1), \dots, \sigma_n(x_n)),$$

   is a solution of the n-SE.
3. If $\varphi \in Sym(X)$ is an arbitrary bijection of the set X onto itself, then $(\varphi)^{\times n} \circ R \circ (\varphi^{-1})^{\times n}$ is a solution of the n-SE.

§ 4. Rational solutions and Tropicalization

Let $\mathbb{R}(x_1, x_2, \dots, x_n)$ be the field of rational fractions.

Any n-tuple $(r_1, r_2, \dots, r_n)$ of rational fractions defines a map

$$R : \mathbb{R}^n \to \mathbb{R}^n$$

by the rule

$$R(x_1, x_2, \dots, x_n) = (r_1, r_2, \dots, r_n).$$

If $(\mathbb{R}, R)$ is a solution of n-SE, then it is called by a rational solution.

Let I_n be a subset on non-zero fractions $r = f/g \in \mathbb{R}(x_1, x_2, \ldots, x_n)$ such that

– all coefficients f are equal to 1 and the free term is equal to 0;

– g is equal to 1 or all its coefficients are equal to 1 and the free term is equal to 0.

Let PL_n be the set of piecewise linear functions $\mathbb{R}^n \to \mathbb{R}$.

Definition

A rational solution of n-SE,

$$R(x_1, x_2, \dots, x_n) = (r_1(x_1, \dots, x_n), r_2(x_1, \dots, x_n), \dots, r_n(x_1, \dots, x_n)),$$

is said to be a I-rational solution if all r_i lie in I_n.

Example

It is easy to see that the famous electric solution of TE,

$$R_E(x,y,z)=\left(\frac{xy}{x+z+xyz},\ x+z+xyz,\ \frac{yz}{x+z+xyz}\right)$$

and the solution that is obtained from R_E by removing terms of degree three,

$$R_e(x,y,z)=\left(\frac{xy}{x+z},\ x+z,\ \frac{yz}{x+z}\right),$$

are I-rational solutions.

Definition

The tropicalization is a function ${}^t : I_n \to PL_n$ that is defined on $r \in I_n$ using the following recursive procedure.

Let r, r_1, r_2 be rational functions from I_n. Then

1. if $r = x_i$, then $r^t = x_i$, for $i = 1, \ldots, n$;
2. $(r_1 + r_2)^t = \max\{r_1^t,\ r_2^t\}$;
3. $(r_1 r_2)^t = r_1^t + r_2^t$;
4. $\left(\frac{r_1}{r_2}\right)^t = r_1^t - r_2^t$.

Definition

Let

$$R(x_1, \dots, x_n) = (r_1(x_1, \dots, x_n), \dots, r_n(x_1, \dots, x_n)) \in I_n^n$$

be a rational vector-valued map of n variables, where $r_1, \dots, r_n$ lie in I_n. Define the tropicalization of the rational map R componentwise:

$$R^t(x_1, \dots, x_n) := (r_1^t(x_1, \dots, x_n), \dots, r_n^t(x_1, \dots, x_n)).$$

Example

The tropicalization of R_E gives

$$R^t_E(x,y,z) = (x+y-M,\ M,\ y+z-M)\,,$$

where $M = \max\{x, z, x+y+z\}$.

The tropicalization of R_e gives

$$R^t_e(x,y,z) = (x+y-\max\{x,z\},\ \max\{x,z\},\ y+z-\max\{x,z\}).$$

One can check that R^t_E and R^t_e are solutions of TE.

Theorem

If

$$(\mathbb{R}_+, R), \quad R \in I_n^n$$

is a I-rational solution of the n-simplex equation, then its tropicalization

$$(\mathbb{R}, R^t)$$

is a piecewise linear solution of the n-simplex equation.

§ 5. Group extensions and parametric Yang-Baxter equation

Let a group G be an extension of H by K,

$$1 \to H \xrightarrow{i} G \xrightarrow{j} K \to 1,$$

M. Preobrazhenskaya and D. Talalaev (2021) in the case of abelian K, construct a solution of a parametric YBE,

$$R_{12}^{a,b} \, R_{13}^{a,c} \, R_{23}^{b,c} = R_{23}^{b,c} \, R_{13}^{a,c} \, R_{12}^{a,b}, \quad a, b, c \in K,$$

on H.

Suppose, that on G is defined a binary algebraic operation

$$* : G \times G \to G$$

such that

1. $(G, *)$ is a right distributive groupoid;
2. H is closed under multiplication $*$;
3. $*$ defines a right distributive groupoid on K.

On the set of pairs

$$(x, a) \in H \times K$$

define the multiplication

$$(x, a) * (y, b) = (x \underset{a,b}{*} y, a * b) \quad \text{for some } x \underset{a,b}{*} y \in H.$$

Hence, on H we have operation $*$ and a set of operations

$$\{ \underset{a,b}{*} \mid a, b \in K \}.$$

The map

$$R(g,h) = (g, h * g), \quad g, h \in G,$$

defines a solution of the YBE on G.

Proposition

If

$$R^{u,v}(x,y) = (x, y \underset{v,u}{*} x), \quad u, v \in K$$

is the parametric map $H \times H \to H \times H$, then for any $a, b, c \in K$ the following equality

$$R_{12}^{a,b}\, R_{13}^{a,c*b}\, R_{23}^{b,c} = R_{23}^{b*a,c*a}\, R_{13}^{a,c}\, R_{12}^{a,b}$$

holds in H.

Corollary

If $(K, *)$ is a trivial right distributive groupoid, i.e. $u * v = u$ for any $u, v \in K$, then for any $a, b, c \in K$ the following equality

$$R_{12}^{a,b} \, R_{13}^{a,c} \, R_{23}^{b,c} = R_{23}^{b,c} \, R_{13}^{a,c} \, R_{12}^{a,b}$$

holds in H.

If we put $g * h = h^{-1} g h$, we get the result of M. Preobrazhenskaya and D. Talalaev.

§ 6. Tetrahedral equation and ternoids

An algebraic system with one ternary operation is called by ternar, an algebraic system with k ternary operations is called by k-ternoid.

If

$$R = (f, g, h) : X^3 \to X^3$$

is a solution of the TE on some set X, then we can define on X three ternar operations

$$[a, b, c] = f(a, b, c), \quad \langle a, b, c\rangle = g(a, b, c), \quad \{a, b, c\} = h(a, b, c), \quad a, b, c \in X.$$

Hence, a solution of TE defines a 3-ternoid.

Proposition

Let $(X, [\cdot,\cdot,\cdot], \langle \cdot,\cdot,\cdot \rangle, \{\cdot,\cdot,\cdot\})$ be a 3-ternoid. Then it defines a solution of the TE if and only if the following equalities hold

$$[[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]] = [[x, y, z], t, p],$$

$$\langle [x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\rangle = [\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle$$

$$\{[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\} =$$

$$= [\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}],$$

$$\langle x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\rangle = \langle \langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\rangle,$$

$$\{x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\} = \langle \{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t,$$

$$\{y, t, \{z, p, q\}\}) = \{\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}\}.$$

for all $(x, y, z, t, p, q) \in X^6$.

Corollary

Let $(X, [\cdot, \cdot, \cdot])$ be a ternar. The map

$$R(a, b, c) = ([a, b, c], b, c), \quad a, b, c \in X.$$

gives a solution of TE if and only if

$$[[x, t, p], [y, t, q], [z, p, q]] = [[x, y, z], t, p], \quad \text{for all } x, y, z, t, p, q \in X.$$

We will call a 4-groupoid $(X, *, \circ, \triangleleft, \triangleright)$ by IE-groupoid if it satisfies the axioms

1) $x \triangleright (y * z) = (x \triangleright y) * (x \triangleright z)$,

2) $(x \circ y) \triangleleft z = (x \triangleleft z) \circ (y \triangleleft z)$,

3) $(x * y) \circ (z * w) = (x \circ z) * (y \circ w)$,

4) $(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z)$,

5) $(x * y) \triangleleft z = x \triangleright (y \circ z)$,

for all $x, y, z, w \in X$.

Proposition

Any IE-groupoid $(X, *, \circ, \triangleleft, \triangleright)$ gives an elementary solution (X, R) of TE if we put

$$R(x, y, z) = (x,\ x \triangleright (y \circ z),\ z), \quad x, y, z \in X.$$

Example

Let V be a vector space, define 4-groupoid $(V, *, \circ, \triangleleft, \triangleright)$ with operations:

$$
\begin{aligned}
x * y &:= (1-\beta)x + \beta y,\\
x \circ y &:= \beta x + (1-\beta)y,\\
x \triangleleft y &:= (1-\beta)x + y,\\
x \triangleright y &:= x + (1-\beta)y,
\end{aligned}
$$

where β is some endomorphism of the vector space V. Then this 4-groupoid is IE-groupoid and gives the solution

$$R(x, y, z) = (x,\ (1-\beta)x + \beta y + (1-\beta)z,\ z).$$

On the other side, suppose that (X, R) is an elementary solution of TE,

$$R(x, y, z) = (x, [x, y, z], z),$$

such that there is $c \in X$ for which $[c, c, c] = c$, and an unary operation $\{\cdot\} : X \to X$,

$$\{[c, x, c]\} = [c, \{x\}, c] = x, \quad \{[x, y, c]\} = [\{x\}, \{y\}, c],$$

$$\{[c, x, y]\} = [c, \{x\}, \{y\}].$$

Proposition

If we put

$$x * y = [x, y, c], \qquad x \circ y = [c, x, y],$$

$$x \triangleright y = [x, \{y\}, c], \qquad x \triangleleft y = [c, \{x\}, y],$$

then we get an IE-groupoid.

Example

If (V, R) is a solution, where V is a vector space and

$$R(x, y, z) = (x,\ (1-\beta)x + \beta y + (1-\beta)z,\ z), \quad \beta \in Aut(V),$$

then by taking $c := 0$ and $\{x\} := \beta^{-1}x$ we get a IE-groupoid.

§ 7. Verbal solutions for the tetrahedral equation

Let G be a group. A verbal solution (G, R) of the n-SE is a solution

$$R(g_1, \dots, g_n) = (w_1(g_1, \dots, g_n), w_2(g_1, \dots, g_n), \dots, w_n(g_1, \dots, g_n)),$$

where $w_i = w_i(x_1, \dots, x_n)$ are reduced words in the free group $F_n = \langle x_1, \dots, x_n \rangle$.

A verbal solution R of the n-SE is said to be l-elementary if it does not fix only l-th component.

For arbitrary group G there is a map $R : G^2 \to G^2$ which is an elementary solution for the YBE.

For example, we can take any quandle on G ($Conj_n(G)$, or $Core(G)$) and construct elementary solution on G.

Question

Let F be a non-abelian free group. Is there a map $R : F^n \to F^n$, $n > 2$, that gives a bijective non-trivial (elementary) solution for n-SE?

By a trivial solution we mean a permutation of components or solution which comes from a solution of $(n-1)$-SE.

In the case $n = 3$ a description of verbal 3-elementary solutions gives

Theorem

Let $R : G^3 \to G^3$ be a verbal 3-elementary solution of TE for every group G, then it has one of the following forms:

1. $R(x, y, z) = (x, y, yx^{-1})$,
2. $R(x, y, z) = (x, y, x^{-1}y)$,
3. $R(x, y, z) = (x, y, w(y, z))$, where $R'(y, z) = (y, w(y, z))$ is a solution of YBE.

Thank you!