Quasiconformal mappings and Neumann eigenvalues of the p-Laplace operator

Valerii Pchelintsev

Tomsk Polytechnic University
Tomsk State University

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[0] This talk is based on joint works with V. Gol'dshtein and A. Ukhlov

Main problem :

Lower estimates of the first non-trivial Neumann eigenvalues of the p-Laplace operator, $1 < p < \infty$:

$$\Delta_p u = -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)), \quad x \in \Omega,$$

in bounded simply connected domains $\Omega \subset \mathbb{R}^2$.

The classical Neumann spectral problem for $\Delta_p u$:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu_p |u|^{p-2}u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The weak statement of this spectral problem : $u \in W^1_p(\Omega)$ solves the previous problem iff

$$\int\limits_{\Omega} (|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x))\,dx = \mu_p \int\limits_{\Omega} |u(x)|^{p-2}u(x)v(x)\,dx$$

for all $v \in W^1_p(\Omega)$.

By the Min–Max Principle the first non-trivial Neumann eigenvalue of the p-Laplace operator $\mu_1(\Omega)$ can be characterized as

$$\mu_1(\Omega) = \min \left\{ \frac{\|\nabla u \mid L_p(\Omega)\|^p}{\|u \mid L_p(\Omega)\|^p} : u \in W_p^1(\Omega) \setminus \{0\}, \int\limits_{\Omega} |u|^{p-2} u \, dx = 0 \right\}.$$

Moreover, $\mu_1(\Omega)^{-\frac{1}{p}}$ is equal to the best constant $B_{p,p}(\Omega)$ in the p-Poincaré-Sobolev inequality

$$\inf_{c \in \mathbb{R}} ||u - c \mid L_p(\Omega)|| \leq B_{p,p}(\Omega) ||\nabla u \mid L_p(\Omega)||, \quad u \in W_p^1(\Omega).$$

The Sobolev space $W_p^1(\Omega)$, $1 \leq p < \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm :

$$\|f \mid W_p^1(\Omega)\| = \left(\int\limits_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left(\int\limits_{\Omega} |\nabla f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = 2$, then the classical upper estimate for the first non-trivial Neumann eigenvalue of the Laplace operator states that :

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^* is a ball of the same volume as Ω.

The classical result by L. E. Payne and H. F. Weinberger (1960) for the Laplace operator states that in convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$,

$$\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},$$

where $d(\Omega)$ is a diameter of a convex domain Ω.

In 2013 it was proved (L. Esposito, C. Nitsch, C. Trombetti) that if $\Omega \subset \mathbb{R}^n$ is a bounded convex domain having diameter d then for $p \geq 2$

$$\mu_1(\Omega) \geq \left(\frac{\pi_p}{d(\Omega)}\right)^p,$$

where

$$\pi_p = 2 \int\limits_0^{(p-1)^{\frac{1}{p}}} \frac{dt}{(1 - t^p/(p-1))^{\frac{1}{p}}} = 2\pi \frac{(p-1)^{\frac{1}{p}}}{p \sin(\pi/p)}.$$

In non-convex domains the first non-trivial Neumann eigenvalues can not be estimated in the terms of Euclidean diameters of domains.

It can be seen by considering a domain consisting of two identical squares connected by a thin corridor :

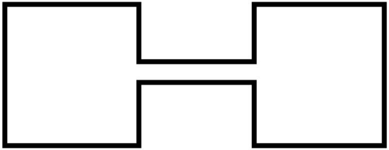

FIGURE: Nikodim-type example.

We suggest the method that is based on the following diagram proposed in (V. Gol'dstein and L. Gurov, 1994) :

$$\begin{array}{ccc} W^1_p(\Omega) & \xrightarrow{\varphi^*} & W^1_q(\mathbb{D}) \\ \downarrow & & \downarrow \\ L_s(\Omega) & \xleftarrow{(\varphi^{-1})^*} & L_r(\mathbb{D}). \end{array}$$

φ^* is a bounded composition operator on Sobolev spaces, $\varphi^*(f) = f \circ \varphi$;

$(\varphi^{-1})^*$ is a bounded composition operator on Lebesgue spaces, $(\varphi^{-1})^*(g) = g \circ \varphi^{-1}$.

In the terms of Sobolev-Poincaré inequalities this diagram can be considered as a change of variables in the inequality :

$$\inf_{c \in \mathbb{R}} ||u - c \mid L_p(\Omega)|| \leq B_{p,p}(\Omega) ||\nabla u \mid L_p(\Omega)||, \quad u \in W^1_p(\Omega).$$

Let Ω and Ω' be domains in $\mathbb{R}^2$. A homeomorphism $\varphi : \Omega \to \Omega'$ is called a K-quasiconformal mapping if $\varphi \in W^1_{2,\mathrm{loc}}(\Omega)$ and there exists a constant $1 \leq K < \infty$ such that

$$|D\varphi(x)|^2 \leq K|J(x,\varphi)| \text{ for almost all } x \in \Omega.$$

The linear case $p = 2$

Estimates of a norm of a composition operator

$$\varphi^* : L^1_2(\Omega) \to L^1_2(\mathbb{D})$$

generated by the composition rule $\varphi^*(f) = f \circ \varphi$ follow from :

S. K. Vodop'yanov and V. Gol'dshtein (1975). *A homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is a K-quasiconformal mapping iff φ generates by the composition rule $\varphi^*(f) = f \circ \varphi$ an isomorphism of Sobolev spaces $L^1_n(\Omega)$ and $L^1_n(\widetilde{\Omega})$:*

$$\|\varphi^*(f) \mid L^1_n(\Omega)\| \leq K^{\frac{1}{n}} \|f \mid L^1_n(\widetilde{\Omega})\|$$

for any $f \in L^1_n(\widetilde{\Omega})$.

Estimates of a norm of a composition operator

$$(\varphi^{-1})^{*} : L_r(\mathbb{D}) \to L_s(\Omega)$$

generated by the composition rule $(\varphi^{-1})^{*}(g) = g \circ \varphi^{-1}$ are based on the notion of quasiconformal regular domains.

We say that a domain $\Omega \subset \mathbb{R}^2$ is called a K-quasiconformal β-regular domain if $\exists$ a K-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$ such that

$$\int\limits_{\mathbb{D}} |J(x,\varphi)|^{\beta}\, dx < \infty \quad \textit{for some} \quad \beta > 1,$$

where $J(x,\varphi)$ is the Jacobian of a mapping φ at a point $x \in \mathbb{D}$.

Lemma 1. *Let Ω be a K-quasiconformal β-regular domain. Then for any function $f \in L_r(\Omega, h)$, $\beta/(\beta-1) \leq r < \infty$, the inequality*

$$||f \mid L_s(\Omega)|| \leq \left(\int_{\mathbb{B}} \left|J(x,\varphi)\right|^{\beta}\, dx\right)^{\frac{1}{\beta}\cdot\frac{1}{s}} ||f \mid L_r(\Omega, h)||$$

holds for $s = \frac{\beta-1}{\beta} r$.

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a K-quasiconformal β-regular domain. Then :*
• *the embedding operator*

$$i_\Omega : W_2^1(\Omega) \hookrightarrow L_s(\Omega)$$

is compact $\forall\ s \geq 1$ *;*
• $\forall\ f \in W_2^1(\Omega)$ *and* $\forall\ s \geq 1$*, the Poincaré–Sobolev inequality*

$$\inf_{c \in \mathbb{R}} \|f - c \mid L_s(\Omega)\| \leq B_{s,2}(\Omega) \|\nabla f \mid L_2(\Omega)\|$$

holds with the constant

$$B_{s,2}(\Omega) \leq K^{\frac{1}{2}} B_{\frac{\beta s}{\beta-1},2}(\mathbb{D}) \|J_\varphi \mid L_\beta(\mathbb{D})\|^{\frac{1}{s}}.$$

where

$$B_{\frac{\beta s}{\beta-1},2}(\mathbb{D}) \leq \left(2^{-1}\pi\right)^{\frac{2-r}{2r}} (r+2)^{\frac{r+2}{2r}}\ .$$

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a K-quasiconformal β-regular domain. Then the spectrum of the Neumann–Laplace operator in Ω is discrete, and can be written in the form of a non-decreasing sequence :*

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,$$

and

$$\frac{1}{\mu_1(\Omega)} \leq \frac{4K}{\sqrt[\beta]{\pi}} \left(\frac{2\beta - 1}{\beta - 1}\right)^{\frac{2\beta-1}{\beta}} \left\| J_\varphi \mid L_\beta(\mathbb{D}) \right\|, \ \beta < \infty.$$

In case $\beta = \infty$

$$\frac{1}{\mu_1(\Omega)} \leq \frac{K}{(j'_{1,1})^2} \left\| J_\varphi \mid L_\beta(\mathbb{D}) \right\|,$$

where $j'_{1,1} \approx 1.84118$ and $\varphi : \mathbb{D} \to \Omega$ is the K-quasiconformal mapping.

Example 1. The homeomorphism

$$w = Az + B\overline{z}, \quad z = x + iy, \quad A > B \geq 0,$$

is K-quasiconformal with $K = \frac{A+B}{A-B}$ and maps the unit disc $\mathbb{D}$ onto the interior of the ellipse

$$\Omega_e = \left\{(u,v) \in \mathbb{R}^2 : \frac{u^2}{(A+B)^2} + \frac{v^2}{(A-B)^2} = 1\right\}.$$

Then by Theorem B in case $\beta = \infty$ we have

$$\mu_1(\Omega_e) \geq \frac{(j'_{1,1})^2}{(A+B)^2}.$$

This estimate is better than the classical estimate for convex domains $\mu_1(\Omega_e) \geq (\pi/d(\Omega_e))^2$, since $d(\Omega_e) = 2(A+B)$ and $2j'_{1,1} > \pi$, $j'_{1,1} \approx 1.84118$.

Example 2. The homeomorphism

$$w = |z|^k z, \quad z = x + iy, \quad k \geq 0,$$

is $(k+1)$-quasiconformal and maps the square

$$Q := \left\{(x,y) \in \mathbb{R}^2 : -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} < y < \frac{\sqrt{2}}{2}\right\}$$

onto star-shaped domains Ω_ε^* with vertices $(\pm\sqrt{2}/2, \pm\sqrt{2}/2), (\pm\varepsilon, 0)$ and $(0, \pm\varepsilon)$, where $\varepsilon = (\sqrt{2}/2)^{k+1}$.

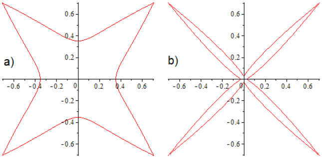

FIGURE: Domains Ω_ε^* under $\varepsilon = \frac{1}{2\sqrt{2}}$ and $\varepsilon = \frac{1}{32}$.

Then by Theorem B in case $\beta = \infty$ we have

$$\mu_1(\Omega_\varepsilon^*) \geq \frac{\pi^2}{2(k+1)^2}.$$

The degenerated case $p > 2$

We estimate a norm of a composition operator

$$\varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D}),$$

generated by the composition rule $\varphi^*(f) = f \circ \varphi$ a K-quasiconformal mapping

$$\varphi : \mathbb{D} \to \Omega.$$

The composition operator is bounded if and only if

$$K_{p,q}(\mathbb{D}) = \left(\int\limits_{\mathbb{D}} \left(\frac{|D\varphi(x)|^p}{|J(x,\varphi)|} \right)^{\frac{q}{p-q}} \, dx \right)^{\frac{p-q}{pq}} < \infty,$$

and the norm of the composition operator $\|\varphi^*\| \leq K_{p,q}(\mathbb{D})$.

In the case of K-quasiconformal mappings $\varphi : \mathbb{D} \to \Omega$, $1 \leq q \leq 2 < p < \infty$

$$K_{p,q}(\mathbb{D}) \leq K^{\frac{1}{2}} \pi^{\frac{2-q}{2q}} |\Omega|^{\frac{p-2}{2p}},$$

Estimates of a norm of a composition operator

$$(\varphi^{-1})^* : L_r(\mathbb{B}) \to L_s(\Omega),$$

generated by the composition rule $(\varphi^{-1})^*(g) = g \circ \varphi^{-1}$ are based on the notion of quasiconformal regular domains.

Theorem C. *Let Ω be a K-quasiconformal β-regular domain, $r = p\beta/(\beta-1)$, $p > 2$. Then the following inequality holds*

$$\frac{1}{\mu_1(\Omega)} \leq$$
$$\inf_{q\in(q^*,2]} \left\{ 2^p \left(\frac{1-\frac{1}{q}+\frac{1}{r}}{\frac{1}{2}-\frac{1}{q}+\frac{1}{r}} \right)^{p-\frac{p}{q}+\frac{p}{r}} \pi^{\frac{p}{r}-\frac{p}{2}} \right\} K^{\frac{p}{2}} |\Omega|^{\frac{p-2}{2}} \cdot ||J_\varphi \mid L_\beta(\mathbb{D})||,$$

where $q^* = 2\beta p/(\beta p + 2(\beta-1))$.

Example 3. For $n \in \mathbb{N}$, the homeomorphism

$$\varphi(z) = A\left(z + \frac{z^n}{n}\right) + B\left(\overline{z} + \frac{\overline{z}^n}{n}\right), \quad z = x + iy, \quad A > B \geq 0,$$

is quasiconformal with $K = (A+B)/(A-B)$ and maps the unit disc $\mathbb{D}$ onto the domain Ω_n bounded by an epicycloid of $(n-1)$ cusps, inscribed in the ellipse with semi-axes $(A+B)(n+1)/n$ and $(A-B)(n+1)/n$.

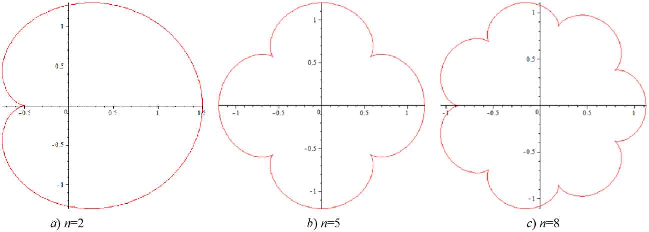

FIGURE: Image of $\mathbb{D}$ under $\varphi(z)$.

Then by Theorem C in case $\beta = \infty$ we have

$$\frac{1}{\mu_1(\Omega)} \leq \inf_{q \in (q^*, 2]} \left\{ \left(\frac{1 - \frac{1}{q} + \frac{1}{p}}{\frac{1}{2} - \frac{1}{q} + \frac{1}{p}} \right)^{p+1-\frac{p}{q}} \right\} 2^{p+2} (A+B)^p \left(\frac{n+1}{n} \right)^{\frac{p}{2}-1},$$

where $q^* = 2p/(p+2)$.

The singular case $1 < p < 2$

We estimate a norm of a composition operator

$$\varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D}),$$

generated by the composition rule $\varphi^*(f) = f \circ \varphi$ a K-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$, using the generalized Brennan's conjecture :

$$\int_{\Omega} |D\varphi(x)|^{\beta}\, dx < +\infty, \quad \text{for all} \quad \frac{4K}{2K+1} < \beta < \frac{4K}{2K-1}.$$

A connection between Brennans conjecture and composition operators on Sobolev spaces :

Theorem. *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected domain. Generalized Brennan's Conjecture holds for a number* $\beta \in (4K/(2K+1), 4K/(2K-1))$ *if and only if any* K*-quasiconformal homeomorphism* $\varphi : \mathbb{D} \to \Omega$ *induces a bounded composition operator*

$$\varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D})$$

for any $p \in (2, +\infty)$ *and* $q = p\beta/(p+\beta-2)$.

In case of K-quasiconformal mappings $\varphi : \mathbb{D} \to \Omega$,

$$K_{p,q}(\mathbb{D}) \leq K^{\frac{1}{p}} \left(\int\limits_{\mathbb{D}} |D\varphi(x)|^{\frac{(p-2)q}{p-q}} dx \right)^{\frac{p-q}{pq}},$$

if

$$\frac{4K}{2K+1} < p < 2,$$

$$1 \leq q < \frac{2p}{4K-(2K-1)p}.$$

Theorem D. *Let Ω be a K-quasiconformal β-regular domain and $\varphi : \mathbb{D} \to \Omega$ be a K-quasiconformal mapping. Suppose that the Brennan's Conjecture holds. Then for any*

$$\frac{4K}{2K+1} < p < 2$$

the following estimate

$$\frac{1}{\mu_1(\Omega)} \leq K \|J_\varphi \mid L_\beta(\mathbb{D})\| \inf_{q \in I} \left\{ \left(\frac{2}{\pi^\nu} \left(\frac{1-\nu}{1/2-\nu} \right)^{1-\nu} \right)^p \| |D\varphi|^{p-2} \mid L_{\frac{q}{p-q}}(\mathbb{D})\| \right\}$$

holds, where $I = [1, 2p/(4K-(2K-1)p))$ and $\nu = 1/q - (\beta-1)/\beta p$.

Example 4. The homeomorphism

$$w = |z|^k z, \quad z = x + iy, \quad k \geq 0,$$

is $(k+1)$-quasiconformal and maps the square

$$Q := \left\{ (x,y) \in \mathbb{R}^2 : -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} < y < \frac{\sqrt{2}}{2} \right\}$$

onto star-shaped domains Ω^*_ε with vertices $(\pm\sqrt{2}/2, \pm\sqrt{2}/2), (\pm\varepsilon, 0)$ and $(0, \pm\varepsilon)$, where $\varepsilon = (\sqrt{2}/2)^{k+1}$.

In the case of porous media flows ($p = 3/2$), taking $q = 1$, we have

$$\frac{1}{\mu^{(1)}_{3/2}(\Omega^*_{\varepsilon(k)})} \leq 16\sqrt{\frac{(k+1)^3}{2-k}}, \; 0 \leq k < 2.$$

Now we precise Theorems B (C, D) in quasidiscs.

K-quasidiscs are images of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ under K-quasiconformal homeomorphisms of the plane $\mathbb{R}^2$.

This class includes all Lipschitz simply connected domains but also includes a class of fractal type domains like snowflakes. The Hausdorff dimension of the quasidisc's boundary can be any number in $[1, 2)$.

The suggested approach is based on the sharp inverse Hölder inequality for Jacobians of quasiconformal mappings.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a K-quasiconformal mapping. Then for every disc $\mathbb{D} \subset \mathbb{R}^2$ and for any $1 < \kappa < \frac{K}{K-1}$ the inverse Hölder inequality

$$\left(\int_{\mathbb{D}} |J(x,\psi)|^{\kappa} \, dx \right)^{\frac{1}{\kappa}} \leq \frac{C_\kappa^2 K \pi^{\frac{1}{\kappa}-1}}{4} \exp\left\{ \frac{K\pi^2(2+\pi^2)^2}{2\log 3} \right\} \int_{\mathbb{D}} |J(x,\psi)| \, dx$$

holds, where

$$C_\kappa = \frac{10^6}{[(2\kappa-1)(1-\nu)]^{1/2\kappa}}, \quad \nu = 10^{8\kappa} \frac{2\kappa-2}{2\kappa-1}(24\pi^2 K)^{2\kappa} < 1.$$

Theorem E. *Let Ω be a K-quasidisc. Then*

$$\mu_1(\Omega) \geq \frac{M_p(K)}{|\Omega|},$$

where $M_p(K)$ depends only on p and the quasiconformity coefficient K of Ω.

Quasiconformal mappings preserving measure

Example 1. The homeomorphism

$$\varphi(z) = \sqrt{a^2+1}z + a\overline{z}, \quad z = x+iy, \quad a \geq 0,$$

is a K-quasiconformal with $K = \frac{\sqrt{a^2+1}+a}{\sqrt{a^2+1}-a}$ and maps the unit disc $\mathbb{D}$ onto the interior of ellipse

$$\Omega_e = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{(\sqrt{a^2+1}+a)^2} + \frac{y^2}{(\sqrt{a^2+1}-a)^2} = 1\right\}.$$

The Jacobian $J(z,\varphi) = |\varphi_z|^2 - |\varphi_{\overline{z}}|^2 = 1$.

Example 2. The homeomorphism

$$\varphi(z) = \sqrt{2}(1+z)^{\frac{3}{4}}(1+\overline{z})^{\frac{1}{4}}, \quad z = x+iy,$$

is a K-quasiconformal with $K = 2$ and maps the unit disc $\mathbb{D}$ onto the interior of the "rose petal"

$$\Omega_p := \left\{(\rho,\theta) \in \mathbb{R}^2 : \rho = 2\sqrt{2}\cos(2\theta), \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\right\}.$$

The Jacobian $J(z,\varphi) = |\varphi_z|^2 - |\varphi_{\overline{z}}|^2 = 1$.

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THANKS