Multidimentional algebraic interpolations

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Standard interpolations

The basic, classical (standard) interpolations include interpolations of Lagrange, Hermite, Newton, etc.
Lagrange: Given the points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and the values $c_j \in \mathbb{C}$, find the polynomial $f(z)$ of degree $m-1$ with the property

$$f(w_j) = c_j, \forall j = 1, \ldots, m.$$

Note that the interpolation polynomial is defined in terms of the polynomial $p(z) = (z - w_1) \cdot \ldots \cdot (z - w_m)$ by the formula

$$f(z) = p(z) \sum_{j=1}^{m} \frac{c_j}{z - w_j} \operatorname{res}_{w_j} \left(\frac{1}{p} \right).$$

Thus, specifying the interpolation nodes as the null set of the ideal $\langle p \rangle$ gives a toolkit for constructing an interpolation polynomial.

Standard interpolations

Hermite: Given the points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and the values $c_{j,k} \in \mathbb{C}$, where $j = 1, \ldots, m, \;\; k = 0, \ldots, \mu_j - 1$ find a polynomial $f(z)$ having at given points given values of derivatives up to orders of $\mu_j - 1$, that is,

$$f^{(k)}(w_j) = c_{j,k}, \forall j = 1, \ldots, m, \forall k = 0, \ldots, \mu_j - 1.$$

In this problem, the corresponding ideal is taken by the generated polynomial

$$p(z) = (z - w_1)^{\mu_1} \cdots (z - w_m)^{\mu_m}$$

Non-standard 1-dimensional

Problem: *Given the complex numbers* $a_{j,k}$ *(*$j = 1, \ldots, m$*;* $k = 0, \ldots, \mu_j - 1$*) and* c*. It is necessary to describe the set of all functions* f *which are analytic in the neighborhood of* $\Omega \subset \mathbb{C}$ *points* $w_1, \ldots, w_m$ *and satisfy the equation:*

$$\sum_{j=1}^{m} \sum_{k=0}^{\mu_j - 1} a_{j,k} f^{(k)}(w_j) = c. \tag{1}$$

(D. Alpay, etc., 2016). Note that if f is a solution of (1), then $f + ph$ is also a solution, where

$$p(z) = \prod_{j=1}^{m} (z - w_j)^{\mu_j}, \quad h \in \mathcal{O}(\Omega).$$

In other words, we can work in the factor ring $\mathcal{O}(\Omega)/\langle p \rangle$ by the ideal generated by the polynomial p.

Noetherian opeators

Definition (Ehrenpreis, Palamodov)

Let $I \subset \mathbb{C}[s_1, \ldots, s_n]$ be a primary ideal. A family of linear differential operators with polynomial coefficients $\partial_\ell(\boldsymbol{s}, D)$, $\ell = 1, \ldots, t$ is called a noetherian operator for I, if the conditions

$$\partial_\ell(\boldsymbol{s}, D)\varphi(\boldsymbol{s})|_{V(I)} = 0, \quad \forall \ell = 1, \ldots, t$$

are necessary and sufficient for the function $\varphi(\boldsymbol{s})$ to belong to ideal I.

Noetherian operators in the one-dimensional case

In the one-dimensional case an arbitrary polynomial has the form:

$$p(s) = (s - w_1)^{\mu_1} \cdot \ldots \cdot (s - w_k)^{\mu_k},$$

and its generated ideal is decomposed into the intersection of primal ones

$$\rho_j = \langle (s - w_j)^{\mu_j} \rangle, \quad j = 1, \ldots, k.$$

A necessary and sufficient condition for a given function φ to belong to the primary component ρ_j is vanishing of φ by the following operators with constant coefficients:

$$\mathcal{L}_{j,0}, \mathcal{L}_{j,1}, \ldots, \mathcal{L}_{j,\mu_j - 1},$$

where $\mathcal{L}_{i,j}[\varphi(s)] = \left. \frac{d^j \varphi}{ds^j} \right|_{s = w_i}$.

Non-standard n-dimensional

Problem (Alpay, Yger; 2019)

Let $\boldsymbol{p}^{-1}(0) = \{w_1, \ldots, w_m\}$ and U be an open subset of $\mathbb{C}^n$ containing $\boldsymbol{p}^{-1}(0)$. Fix $a_{j,l}, j = 1, \ldots, m, l \in A_{w_j}$ and c; all of them are complex numbers. We need to describe the space of holomorphic functions $f\colon U \to \mathbb{C}$ with the following property:

$$\sum_{j=1}^{m} \sum_{\ell \in A_{w_j}} a_{j,\ell} \mathcal{L}_{w_j,\ell}[f](w_j) = c. \tag{2}$$

Basis in $\mathbb{C}[s]/\langle p\rangle$

The following monomial basis

$$\mathcal{B} = \{\boldsymbol{s}^{\beta_k}; k = 0, \ldots, N(\boldsymbol{p}) - 1\}$$

in the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p\rangle$ is one of ingredients for solving the interpolation problem. In fact, this factor is the space of reminders when dividing polynomials by the ideal $\langle \boldsymbol{p}\rangle$. The basis $\mathcal{B}$ is constructed using the Gröbner basis for the ideal $\langle \boldsymbol{p}\rangle$.

Solution of the multidimensional Problem

Theorem (Alpay, Yger)

Let $\{w_1, \ldots, w_m\} = \boldsymbol{p}^{-1}(0)$, U *be an open subset in* $\mathbb{C}^n$ *containing* $\boldsymbol{p}^{-1}(0)$. *Let the sequence*

$$\boldsymbol{a} = \{a_{j,\boldsymbol{\ell}}, j = 1, \ldots, m, \boldsymbol{\ell} \in A_{w_j}\}$$

and the complex number c *be given. Let us denote the polynomials*

$$h^{\boldsymbol{a}}_{w_j}(\boldsymbol{s}) = \sum_{\boldsymbol{\ell} \in A_{w_j}} a_{j,\boldsymbol{\ell}}(\boldsymbol{s} - w_j)^{\boldsymbol{\ell}} / \boldsymbol{\ell}!,$$

making up the sequence $\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}} = [h^{\boldsymbol{a}}_{w_1}, \ldots, h^{\boldsymbol{a}}_{w_m}]$, *and let*

$$\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] = (\alpha_0[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}], \ldots, \alpha_{N(\boldsymbol{p})-1}[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}])$$

be the projection of this sequence onto the quotient space $\mathbb{C}[\boldsymbol{z}]/\langle p \rangle$.

Solution of the multidimensional problem

Theorem (Alpay, Yger 2019)

Then:

- *If* $\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] = 0$, *then the problem has no solution in the case* $c \neq 0$, *and any function* $f \in \mathcal{O}(U)$ *is a solution in the case* $c = 0$;
- *If* $\alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}] \neq 0$, *then* $f \in \mathcal{O}(U)$ *satisfies the condition* (2) *iff*

$$\alpha[\boldsymbol{f}] \cdot \boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] \cdot \alpha[\boldsymbol{h}^{\boldsymbol{a}}_{\boldsymbol{w}}]^T = c,$$

where T *is the transposition sign, and* $\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}]$ *is the Grothendieck global residues matrix:*

$$\boldsymbol{Q}_{\boldsymbol{p}}[\mathcal{B}] = \operatorname{Res}\left[\frac{\boldsymbol{s}^{\beta_{k_1}+\beta_{k_2}}\, d\boldsymbol{s}}{p_1(\boldsymbol{s})\ldots p_n(\boldsymbol{s})}\right]_{0\leq k_1,k_2\leq N\langle\boldsymbol{p}\rangle-1}$$

In one variable there are two equivalent definitions of the residue: by integral over small circle

$$\operatorname*{res}_a g = \frac{1}{2\pi i} \int\limits_{|z-a|=\varepsilon} g(z)dz$$

and by coefficient c_{-1} of the Laurent decomposition

$$g(z) = \sum_{k \in \mathbb{Z}} c_k (z-a)^k$$

If a multidimentional analogue of a holomorphic function is understood as a mapping $\mathbb{C}^n \to \mathbb{C}^n$, it is convienient to use the so called *local (pointed) Grothendieck residue* as an integral definition.

Grothendieck residue

Grothendieck residue is a cornerstone of complex analysis and algebraic geometry and it plays a key roles in the theory of singularity and foliations.
Assume that polynomials

$$f_1, \dots, f_n \in \mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_n]$$

have isolated common zero $a \in \mathbb{C}^n$. Consider a rational differential n-form

$$\omega = \frac{1}{(2\pi i)^n} \frac{h(z)\, dz}{f_1(z) \dots f_n(z)}, \quad (\text{with } dz = dz_1 \wedge \dots \wedge dz_n)$$

Grothendieck residue

Definition

The Grothendieck residue, associated with $f = (f_1, \ldots, f_n)$ and h, is determined as an integral

$$\operatorname*{res}_{a}\,_{f}(h) = \int\limits_{\Gamma_a} \omega$$

of the form ω over a very special cycle

$$\Gamma_a = \{z \in U_a \colon |f_j(z) = \varepsilon_j, j = 1, \ldots, n\}$$

where the neighborhood U_a of a and ε_j are chosen such that the closure $\overline{U}_a$ does not contain roots different from a and $\Gamma_a \subset\subset U_a$.

Another integral representation

Our goal is to represent the class of homology of cycle Γ_a as a linear combination of toric cycles. We need the following theorem. Consider the standard differential $(0, n-1)$ form

$$\beta = (n-1)! \frac{\sum_{k=1}^{n} \overline{f}_k d\overline{f}_1 \wedge \dots [k] \dots \wedge d\overline{f}_n}{(|f_1|^2 + \dots + |f_n|^2)^n}$$

Theorem

The local Grothendieck residue admits an integral representation

$$\operatorname*{res}_a{}_f(h) = \int\limits_{S^{2n-1}} h \cdot \beta \, dz,$$

where S^{2n-1} is a sphere arround a of a small radius.

Amoeba and its complement

Definition

Given a Laurent polynomial f its amoeba A_f is the image of the hypersurface $V = f^{-1}(0)$ under the map

$$\mathrm{Log}\colon (z_1, \ldots, z_n) \to (\log|z_1|, \ldots, \log|z_n|).$$

For the amoeba we will also use notation A_V.

Amoeba reflects the distribution of the algebraic set V. But more precisely, the amoeba defines «emptness» for V.

Theorem (Gelfand, Kapranov, Zelevinsky)

The connected components of the amoeba complement ${}^c A_f$ *are convex, and they are in bijective correspondence with the different Laurent expansions (centered at the origin) of the rational function* $1/f$.

The convexity here follows from the general fact that the domains of convergence of Laurent series are exactly the logarithmic convex ones.

Newton polytope of f

The shape of the amoeba is closely related to the Newton polytope Δ_f of the polynomial f. Recall that Δ_f is defined as the convex hull in $\mathbb{R}^n$ of the index set A in the experession

$$f(z_1, \ldots, z_n) = \sum_{\alpha \in A} a_\alpha z^\alpha$$

The set of integer points in Δ_f admits a natural partition $\Delta_f \cap \mathbb{Z}^n = \bigcup_\Gamma A_\Gamma$, where Γ is any face on Δ_f and A_Γ denotes the intersection of $\mathbb{Z}^n$ with the reflective interior of Γ. We shall consider the dual cone C_ν of Δ_f at ν defined as

$$C_\nu = \left\{ s \in \mathbb{R}^n \colon \langle s, \nu \rangle = \max_{\alpha \in \Delta_f} \langle s, \alpha \rangle \right\}$$

Notice that $\dim C_\nu = n - \dim \Gamma$ when $\nu \in A_\Gamma$. In particular, C_ν has nonempty interior if ν is a vertex of Δ_f, and it equals $\{0\}$ whenever ν is an interior point of Δ_f.

The order map on the complement ${}^cA_f$

Theorem (Forsberg, Passare, Tsikh)

On the set $\{E\}$ of connected components of ${}^cA_f$ there is an injective map (the order map)

$$\nu\colon \{E\} \to \Delta_f \cap \mathbb{Z}^n$$

with the property that the dual cone $C_{\nu(E)}$ is equal to the recession cone of E. That is, for any $u \in E$ one has $u + C_\nu \in E$ and no strictly larger cone is contained in E. (Notice that if ν is the k-skeleton of Δ_f the C_ν has dimension $n - k$).
Thus, connected components can be numbered as E_ν with integer $\nu \in \Delta_f$.

Corollary

The cardinality of the set $\{E\}$ of connected components satisfies the inequalities

$$\#\operatorname{Vert}\Delta_f \leqslant \#\{E\} \leqslant \#\Delta_f \cap \mathbb{Z}^n$$

Gelfond-Khovanskii formula

Theorem (Gelfond-Khovanskii formula)

Assume that all faces of the Minkovskii sum $\Delta = \Delta_1 + \cdots + \Delta_n$ of Newton polytopes of polynomials $f_1, \ldots, f_n$ are locked. Then the sum of all local residues in $(\mathbb{C} \setminus 0)^n$ is calculated by the formula:

$$\sum_{\{a\}} \operatorname{res}_{a} {}_{f}(h) = \sum_{\nu \in \operatorname{Vert} \Delta} k_j \operatorname{Res}_{E_\nu} \left(\frac{h}{f_1 \ldots f_n} \right)$$

where $\operatorname{Res}_{E_\nu}$ is the coefficient c_{-I} of the Laurent decomposition for $\frac{h}{f_1 \ldots f_n}$ in the connected component E_ν.

In fact one can prove that the sum $\sum\limits_{\{a\}} \Gamma_a$ of local Grothendieck cycles Γ_a is homologically equivalent to the sum

$$\sum_{\nu \in \operatorname{Vert} \Delta} k_\nu \operatorname{Log}^{-1}(u_\nu), \quad u_\nu \in E_\nu$$

$\mathcal{U}$-resolutions

Let $U = \{U_i\}_{i \in I}$ be an open finite covering of the manifold M. A U-chain of M of degree q and dimension p is an alternating function γ on I^{q+1} to $C_p(M)$ such that

$$\text{support}\,\gamma(i_0, i_1, \ldots, i_q) \subseteq U_{i_0} \cap \cdots \cap U_{i_q}$$

for all $i_0, i_1, \ldots, i_q$.

Definition (Gleason)

Let $\xi \in Z_r(M)$. A U-resolution of ξ is a sequence $\xi_0, \xi_1, \ldots, \xi_r$ such that

(a) ξ_q is a U-chain of degree q and dimension $r - q$

(b) $\xi = \sum_{i \in I} \xi_0(i)$

(c) $\partial \xi_q(i_0, \ldots, i_q) = \sum_{j \in I} \xi_{q+1}(j, i_0, \ldots, i_q)$

Main result

Theorem (Durakov, Tsikh, Ulvert)

Assume that $f = (f_1, \ldots, f_n)$ *has a finite number of solutions in* $(\mathbb{C} \setminus 0)^n$. *Then*

$$\sum_a \operatorname*{res}_{\substack{a \\ f}}(h) = \sum_{\nu \in \partial\Delta \cap \mathbb{Z}^n} k_j \operatorname{Res}_{E_\nu}\left(\frac{h}{f_1 \ldots f_j}\right)$$

In the homological sense it means that

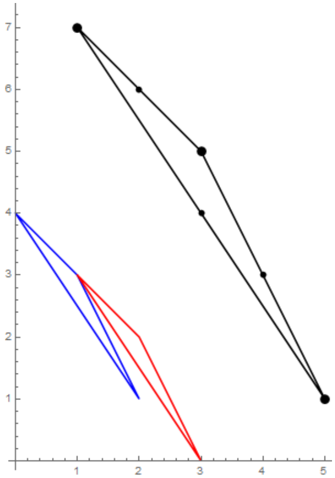
$$\sum_{\{a\}} \Gamma_a = \sum_{\nu \in \partial\Delta \cap \mathbb{Z}^n} k_\nu \operatorname{Log}^{-1}(u_\nu), \quad u_\nu \in E_\nu.$$

Nongeneral position of $\Delta_1, \ldots, \Delta_n$

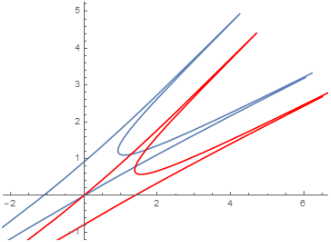
Let us consider the system of polynomials in two variables:

$$f_1 = 3z_1^2 z_2 + z_2^4 + 2z_1 z_2^3,$$
$$f_2 = z_1^3 + 4z_1 z_2^3 + 3z_1^2 z_2^2$$

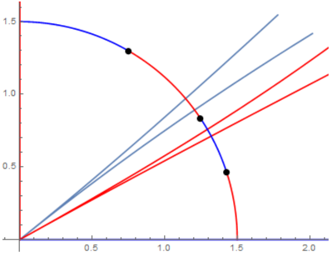
with the following Newton polytopes in nongeneral position.



Nongeneral position of $\Delta_1, \ldots, \Delta_n$



Amoebas A_{f_1} and A_{f_2}



The local distribution at $z = 0$ on the Reinhardt diagram

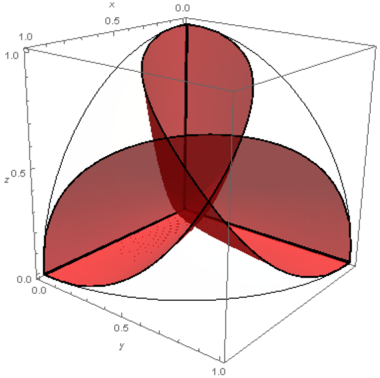
The $\mathcal{U}$-resolution of the sphere S^3 is $\Gamma_{51} - \Gamma_{34} + \Gamma_{13}$ since it is the boundary $\partial\sigma_2$ of union of red arcs. So we get the representation of the Grothendieck cycle by 3 toric cycles, where Γ_{34} does not correspond to a vertex of Δ.

Computation of $\mathcal{U}$-resolution

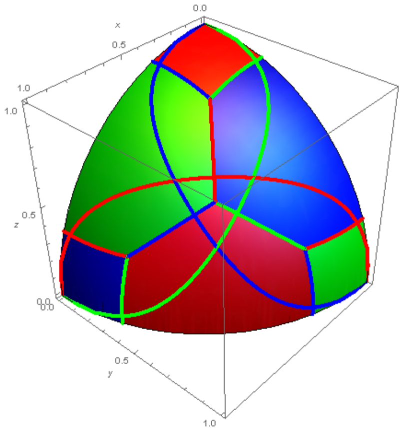
Here on the Reinhardt diagram are 3 surfaces:

$$V_b = \{z_1^3 - z_2 z_3 = 0\},$$
$$V_g = \{z_2^3 - z_1 z_3 = 0\},$$
$$V_r = \{z_3^3 - z_1 z_2 = 0\}$$

distributed in a ball B. We have to construct the $\mathcal{U}$-resolution of the sphere $S^5 \in H_5(B \setminus \{0\})$, respective to the covering $U_b = B \setminus V_b$, $U_g = B \setminus V_g$, $U_r = B \setminus V_r$.



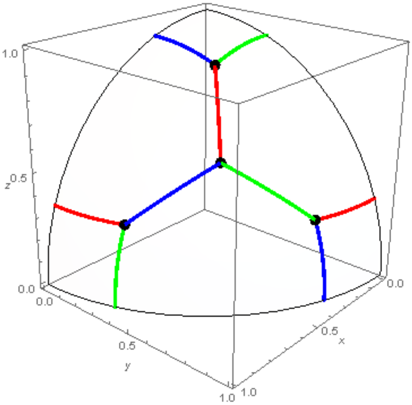
Computation of $\mathcal{U}$-resolution



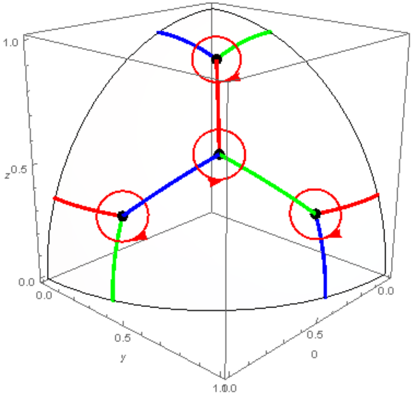
1^{st} step: Decomposition
$S^5 = \sigma_b + \sigma_g + \sigma_r$ by blue, green and red chains. Each of them lies outside of the surface of the corresponding colour, i.e.
$\sigma_b \in U_b$, $\sigma_g \in U_g$, $\sigma_r \in U_r$.

Computation of $\mathcal{U}$-resolution

2^{nd} step: Compute the boundaries $\partial\sigma_b, \partial\sigma_g, \partial\sigma_\nu$. Each edge lies outside of two surfaces with the complement colour.



Computation of $\mathcal{U}$-resolution



3[rd] step: Compute the boundaries of edges taking into account the ordering b, g, ν colours. So we get the resolution ξ as the linear combination

$$\xi = \Gamma_{511} + \Gamma_{115} + \Gamma_{151} - \Gamma_{222}$$

of toric cycles.

The set of differential operators:

$$\{\boldsymbol{\mathcal{L}}_{0,\ell}\} = \Bigg\{\boldsymbol{\mathcal{L}}_{0,000} = \left(-\partial^0 - \frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} - \frac{1}{4!}\frac{\partial^4}{\partial z_1^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_2^4} - \frac{1}{4!}\frac{\partial^4}{\partial z_3^4}\right);$$

$$\boldsymbol{\mathcal{L}}_{0,100} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_1^3} - \frac{\partial^2}{\partial z_2 \partial z_3}\right); \boldsymbol{\mathcal{L}}_{0,010} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_2^3} - \frac{\partial^2}{\partial z_1 \partial z_3}\right);$$

$$\boldsymbol{\mathcal{L}}_{0,001} = \left(-\frac{1}{3!}\frac{\partial^3}{\partial z_3^3} - \frac{\partial^2}{\partial z_1 \partial z_2}\right); \boldsymbol{\mathcal{L}}_{0,110} = \left(-\frac{\partial}{\partial z_3}\right); \boldsymbol{\mathcal{L}}_{0,101} = \left(-\frac{\partial}{\partial z_2}\right);$$

$$\boldsymbol{\mathcal{L}}_{0,011} = \left(-\frac{\partial}{\partial z_1}\right); \boldsymbol{\mathcal{L}}_{0,200} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_1^2}\right); \boldsymbol{\mathcal{L}}_{0,020} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_2^2}\right);$$

$$\boldsymbol{\mathcal{L}}_{0,002} = \left(-\frac{1}{4}\frac{\partial^2}{\partial z_3^2}\right); \boldsymbol{\mathcal{L}}_{0,111} = \left(-\partial^0\right); \boldsymbol{\mathcal{L}}_{0,300} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_1}\right);$$

$$\boldsymbol{\mathcal{L}}_{0,030}\left(-\frac{1}{3!}\frac{\partial}{\partial z_2}\right); \boldsymbol{\mathcal{L}}_{0,003} = \left(-\frac{1}{3!}\frac{\partial}{\partial z_3}\right); \boldsymbol{\mathcal{L}}_{0,400} = \left(-\frac{1}{4!}\partial^0\right);$$

$$\boldsymbol{\mathcal{L}}_{0,040} = \left(-\frac{1}{4!}\partial^0\right); \boldsymbol{\mathcal{L}}_{0,004} = \left(-\frac{1}{4!}\partial^0\right)\Bigg\}.$$

is the standard collection of Noether operators for the ideal:

$$I_0\langle \boldsymbol{p}\rangle = \{(z_1^3 - z_2 z_3)h_1 + (z_2^3 - z_1 z_3)h_2 + (z_3^3 - z_1 z_2)h_3\},$$

where $h_1, h_2, h_3 \in \mathcal{O}_0$.

Theorem

If $\boldsymbol{\alpha}[\boldsymbol{h}_{\boldsymbol{w}}^{\boldsymbol{a}}] \neq 0$, then the holomorphic function $f(\boldsymbol{s})$ satisfies the Alpay-Yger problem for single point ($m = 1$) iff the coordinatization of f satisfies the following condition:

$$\left(a_{000} + a_{111} - \frac{a_{400} + a_{040} + a_{004}}{24}\right)\alpha_1[f] + \left(a_{011} + \frac{a_{300}}{6}\right)\alpha_2[f] +$$
$$+ \left(a_{101} + \frac{a_{030}}{6}\right)\alpha_3[f] + \left(a_{110} + \frac{a_{003}}{6}\right)\alpha_4[f] + \frac{a_{200}}{2}\alpha_5[f] +$$
$$\frac{a_{020}}{2}\alpha_6[f] + \frac{a_{002}}{2}\alpha_7[f] + a_{001}\alpha_8[f] + a_{010}\alpha_9[f] +$$
$$+ a_{001}\alpha_{10}[f] + a_{000}\alpha_{11}[f] = -c.$$

This means that the coordinate vector of f in the local algebra lies in the prescribed affine hyperplane $\Pi_a \subset \mathbb{C}^{11}$.

Theorem

Assume that the system of n germs $p_1, \ldots, p_n \in \mathcal{O}_0$ satisfies the following conditions:

$$\frac{\partial^{\boldsymbol{\alpha}} p_i}{\partial z^{\boldsymbol{\alpha}}}(0) = 0 \qquad 0 \leq \alpha_i \leq d_i - 1 \qquad i = 1, \ldots, n$$

$$\det \left[\frac{\partial^{d_j} p_i}{\partial z_j^{d_j}}(0) \right] \neq 0$$

Then the following equation of ideals holds:

$$\langle p_1, \ldots, p_n \rangle = \langle z_1^{d_1}, \ldots, z_n^{d_n} \rangle$$

A standard interpolation

Let f be the Hermite interpolating polynomial which solves the problem: for given points $\{w_j\}_{j=1}^m \subset \mathbb{C}$ and values $\{c_{j,\ell}\}, j = 1, \ldots, m;\ \ell = 0, \ldots, \mu_j - 1$ find a polynomial $f(z)$ with the property $f^{(\ell)}(w_j) = c_{j,\ell}$. Write f in the form

$$f(z) = \sum_{j=1}^{m} \sum_{\ell=0}^{\mu_j - 1} c_{j,\ell} H_{w_j,\ell}(z).$$

Consider a multidimentional analogues of this problem. Let $p = (p_1, \ldots, p_n)$ be a sequence of polynomials in $z = (z_1, \ldots, z_n)$ with a finite number of roots $p^{-1}(0) = \{w_1, \ldots, w_m\} \in \mathbb{C}^n$. Let B_j be the monomial basis of the local algebra $\mathcal{O}_{w_j}/\langle p \rangle$. The dimention of this algebra is equal to the multiplicity of p at the root w_j.

A standard interpolation

Consider the problem: *For the given* $\{w_1, \ldots, w_m\}$ *and values* $\{c_{j,\ell}\}, j = 1, \ldots, m; \ell \in A_{w_j}$ *(where* $A_{w_j} \subset \mathbb{Z}^n$ *is the set of exponents of basic monomials in* B_j*), find a polynomial* $f(z)$ *with the property*

$$\frac{\partial^{|\ell|} f}{\partial z^{\ell}}(w_j) = c_{j,\ell},$$

where $\ell = (\ell_1, \ldots, \ell_n)$ *and* $|\ell| = \ell_1 + \ldots + \ell_n$*.*

A standard interpolation

Theorem

Assume that for each root $w = w_j \in p^{-1}(0)$ there exist a such vector $d_w \in \mathbb{Z}_+^n$ that

$$\frac{\partial^{|\alpha|} p_i}{\partial z^\alpha}(w) = 0,\ 0 \le \alpha_1 \le d_{1,w}-1, \ldots, 0 \le \alpha_n \le d_{n,w}-1,\ i = 1, \ldots, n,$$

$$\det \left\| \frac{\partial^{d_{k,w}} p_i}{\partial z_k^{d_{k,w}}} \right\| \neq 0 \quad (\textit{here } i, k = 1, \ldots, n).$$

Then the polynomial

$$f(z) = \sum_{j=1}^{m} \sum_{\ell \in A_{w_j}} c_{j,\ell} \left(\prod_{k=1}^{n} H_{j,\ell_k} \right)$$

solves the standard problem

Thank you for your attention!