Lower bounds for the complexity of genuine and virtual 3-manifolds

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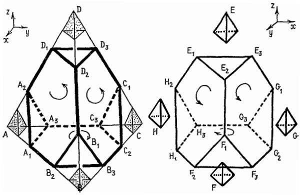
The tetrahedral complexity $c_{tet}(M)$ of a compact 3-manifold M with $\partial M \neq \emptyset$ is equal to k if

- M admits a truncated triangulation with k tetrahedra;
- there is no truncated triangulation of M into fewer tetrahedra.

4_1



ABD – FEH, BCD – EFG,
ADC – EGH, ACB – FHG

Burton – Callahan – Hildebrand – Thistlethwaite – Weeks census, $c_{tet} \leqslant 9$

	Manifolds			Minimal triangulations		
Tetrahedra	Orbl	Non-orbl	Total	Orbl	Non-orbl	Total
1	0	1	1	0	1	1
2	2	2	4	2	3	5
3	9	7	16	10	11	21
4	56	26	82	75	60	135
5	234	78	312	360	179	539
6	962	258	1 220	1 736	801	2 537
7	3 552	887	4 439	7 413	3 202	10 615
8	12 846	2 998	15 844	30 450	12 777	43 227
9	44 250	9 788	54 038	122 136	49 896	172 032
Total	61 911	14 045	75 956	162 182	66 930	229 112

Lower bounds $c_{tet}(M) \geqslant \dfrac{vol(M)}{v_{tet}}$,

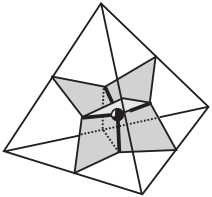
$c = 2$ [M. Fujii, 1990]

8 manifolds

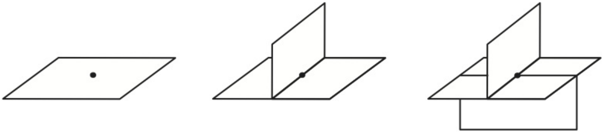
- the same $vol \approx 6.451998$.
- this vol is minimal among all hyperbolic manifolds with totally geodesic boundary.

$3 \leq c \leq 4$ [R. Frigerio – B. Martelli – C. Petronio, 2004]

- 150 manifolds of complexity 3.
- 5,002 manifolds of complexity 4.



- For each tetrahedron Δ of $\mathcal{T}$ consider the union P_Δ of the links of all four vertices of Δ in the first barycentric subdivision.
- $P_{\mathcal{T}} = \bigcup_{\Delta} P_\Delta$.



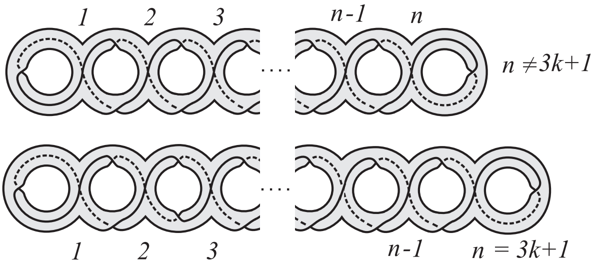
(1) (2) (3)

- Denote by $t_{\mathcal{T}}$ the number of tetrahedra and by $e_{\mathcal{T}}$ the number of edges in $\mathcal{T}$.
- $\chi(M) = \chi(P_{\mathcal{T}}) = e_{\mathcal{T}} - t_{\mathcal{T}}$.

R. Frigerio – B. Martelli – C. Petronio, 2003

Assume that a compact 3-manifold M with $\partial M \neq \emptyset$ admits a truncated triangulation $\mathcal{T}$ with $e_{\mathcal{T}} = 1$. Then

- M is hyperbolic with geodesic boundary.
- $c_{tet}(M) = t_{\mathcal{T}}$.



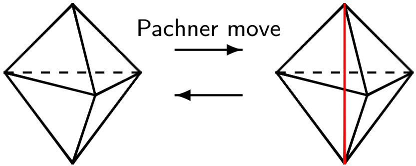
Exact values of $c_{tet}(M)$: manifolds with $e_{\mathcal{T}} = 2$

V. Turaev – A. Vesnin – E. F., 2016

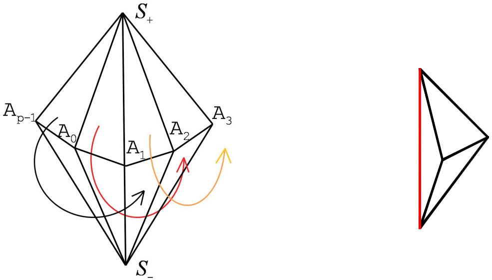
Let a compact 3-manifold M with $\partial M \neq \emptyset$ admit a truncated triangulation $\mathcal{T}$ such that

- $e_{\mathcal{T}} = 2$,
- $\mathcal{T}$ does not admit a $3 \to 2$ Pachner move.

Then $c_{tet}(M) = t_{\mathcal{T}}$.



- Let $n \geq 3$, $0 < k < n$, and $gcd(n; 2-k) = 1$.
- Take n-gonal bipyramid. For every i glue faces $A_iA_{i+1}S_+$ and $S_-A_{i+k}A_{i+k+1}$.



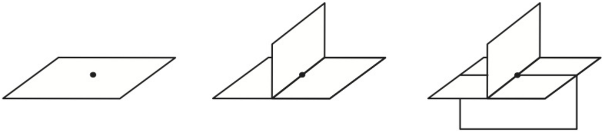
- The result is a pseudo-manifold $N_{n,k}$ with $\chi(N_{n,k}) = n - 1$.
- Cutting a neighborhood of the singular point we get a hyperbolic manifold $M_{n,k}$ with totally geodesic boundary.

Theorem:

If $\partial M \neq \emptyset$, then $c_{tet}(M) \geq rk(H_2(M, \mathbb{Z}_2)) - \chi(M) + 1$.

Proof

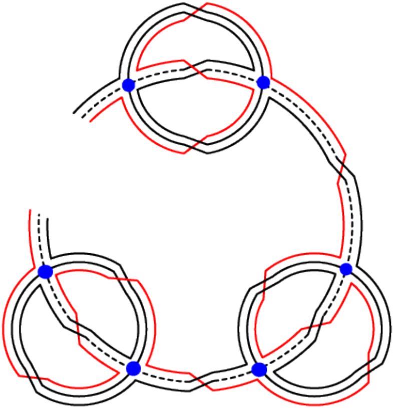
- Let $\mathcal{T}$ be a truncated triangulation of M and $P_{\mathcal{T}}$ be its dual polyhedron.
- $H_2(M, \mathbb{Z}_2) = H_2(P_{\mathcal{T}}, \mathbb{Z}_2)$. Any homology class of $H_2(P_{\mathcal{T}}, \mathbb{Z}_2)$ can be presented by an embedded closed surface in $P_{\mathcal{T}}$.
- $rk(H_2(M, \mathbb{Z}_2)) \leq e_{\mathcal{T}} - 1$.
- $t_{\mathcal{T}} = e_{\mathcal{T}} - \chi(M) \geq rk(H_2(M, \mathbb{Z}_2)) - \chi(M) + 1$.



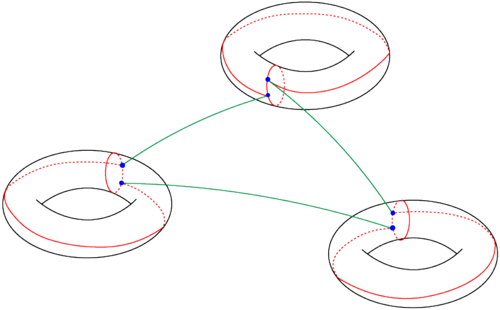
(1) (2) (3)

Exact values of $c_{tet}(M)$: manifolds with $e_{\mathcal{T}} = rk(H_2(P_{\mathcal{T}}, \mathbb{Z}_2)) + 1$

Example 1:



Example 2:



How to describe all manifolds with $e_{\mathcal{T}} = rk(H_2(P_{\mathcal{T}}, \mathbb{Z}_2)) + 1$?

For each 2-component ξ of a special polyhedron P, there is a characteristic map $f : D^2 \to P$, which carries the interior of the disc D^2 onto ξ homeomorphically and which restricts to a local embedding on $S^1 = \partial D^2$. We will call the curve $f|_{\partial D^2} : \partial D^2 \to P$ (and its image $f|_{\partial D^2}(\partial D^2)$) the *boundary curve* $\partial\xi$ of ξ.

It follows that we have exactly two possibilities :

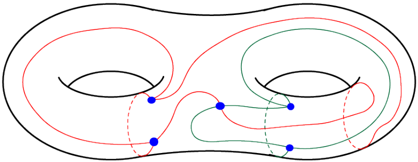
I: $P_{\mathcal{T}}$ has exactly three 2-components and the boundary curve of each of them passes exactly once along each edge of $P_{\mathcal{T}}$.

II: There is exactly one 2-component whose boundary curve passes along each edge of $P_{\mathcal{T}}$ an odd number of times.

Let Φ_1,..,Φ_{d-1} be closed connected surfaces such that $\chi(\Phi_i) \leqslant 0$.

For each Φ_i let us fix a graph $\Gamma_i \subset \Phi_i$ such that:

- Γ_i has at least two vertices of valence 3 and some vertices of valence 4.
- $\Phi_i \setminus \Gamma_i$ is an open disc.



Construction

Let Γ be a graph with vertices of valence 1 and 4 only such that:

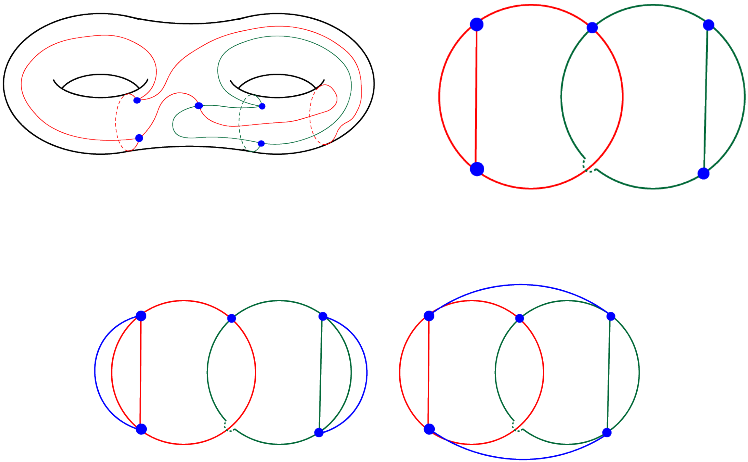
- each connected component of Γ contains at least one vertex of valence 1.
- the number of valence 1 vertices in Γ is equal to the number of all valence 3 vertices in all the graphs $\Gamma_1, \ldots, \Gamma_{d-1}$.

Let ϕ be a bijection between the set of all valence 1 vertices in Γ and the set of all valence 3 vertices in all the graphs Γ_1,..,Γ_{d-1}.

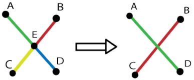
Construct a new graph $\Delta(\phi)$ from the graphs Γ and $\Gamma_1, \ldots, \Gamma_{d-1}$ via the identification map ϕ.

Construct also a new polyhedron P_ϕ from the surfaces Φ_1,..,Φ_{d-1} and the graph Γ via the identification map ϕ.

Obstruction



Преобразуем граф $\Delta(\phi)$ в граф $\tilde{\Delta}(\phi)$ следующим образом: последовательно будем удалять вершины степени 4, лежащие на поверхностях $\Phi_1,..,\Phi_{d-1}$, и склеивать инцидентные ей ребра в графе $\Delta(\phi)$.



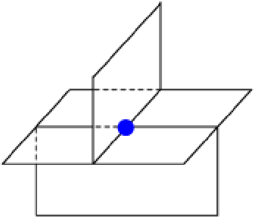
Критерий наличия препятствия

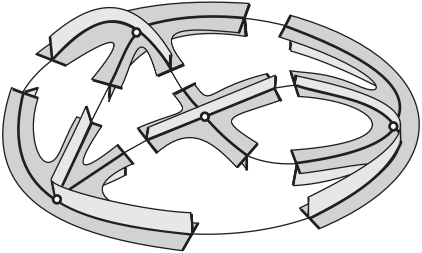
К полиэдру P_1 можно приклеить ровно одну 2-клетку так, чтобы получился специальный полиэдр, тогда и только тогда, когда граф $\tilde{\Delta}(\phi)$ связный.

Лемма

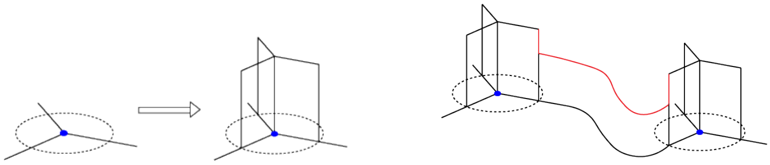
Для любого набора поверхностей $\Phi_1,..,\Phi_{d-1}$ и графов Γ, $\Gamma_1,..,\ \Gamma_{d-1}$ $\exists\phi$ такая, что граф $\tilde{\Delta}(\phi)$ связный.

Поместим в каждую вершину степени 4 графа Γ бабочку.





Возьмем вершину степени 3 на поверхности Φ_i. Часть графа Γ_i, попадающего в достаточно маленькую окрестность вершины представляет из себя триод T. Приклеим к нему $T \times [0,1]$.



Лемма

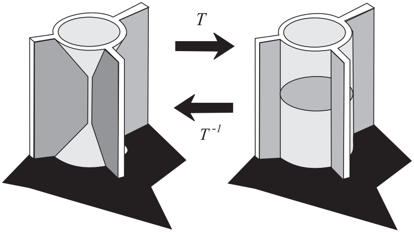
Склейку триодов на ребрах графа Γ можно задать так, что в результате получится полиэдр P_2 с одной компонентой края.

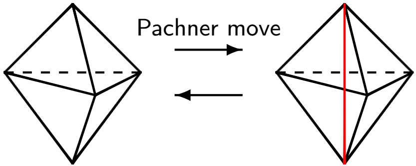
Заклеим край полиэдра P_2 диском и получим требуемый специальный полиэдр P.

Теорема (С. Иванов – Д. Нигомедьянов – Е. Ф., 2019)

Любой специальный полиэдр максимального гомологического ранга, имеющий тип II, можно построить описанным выше способом.

We say that two special polyhedra are equivalent if one of them can be transformed into the other one by a finite sequence of moves $T^{\pm 1}$.





A virtual 3-manifold is the equivalence class $[P]$ of a special polyhedron P.

The complexity $\mathrm{cv}[P]$ of a virtual 3-manifold $[P]$ is equal to k if

- $[P]$ contains a special polyhedron with k true vertices and
- $[P]$ contains no special polyhedra with a smaller number of true vertices.

Thank you!