Quandle rings

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Tomsk
December 10-15, 2019

A quandle is a set with a binary operation that satisfies three axioms motivated by the three Reidemeister moves of diagrams of knots in the Euclidean space $\mathbb{R}^3$. Ignoring the first Reidemeister move gives rise to a weaker structure called a rack.

These algebraic objects were introduced independently by S. V. Matveev and D. Joyce in 1982. They associated a quandle to each tame knot in $\mathbb{R}^3$ and showed that it is a complete invariant up to orientation.

Over the years, racks, quandles and their analogues have been investigated as purely algebraic objects.

A rack is a non-empty set X with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

(R1) For any $x, y \in X$ there exists a unique $z \in X$ such that $x = z * y$;

(R2) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

A rack is called a quandle if the following additional axiom is satisfed:

(Q1) $x * x = x$ for all $x \in X$.

The axioms (R1), (R1) and (Q1) are collectively called quandle axioms.

Example

- If G is a group, then the set G equipped with the binary operation $a * b = b^{-1}ab$ gives a quandle structure on G, called the conjugation quandle, and denoted by $\mathrm{Conj}(G)$.
- If A is an additive abelian group, then the set A equipped with the binary operation $a * b = 2b - a$ gives a quandle structure on A, denoted by $T(A)$ and called the Takasaki quandle of A. For $A = \mathbb{Z}/n\mathbb{Z}$, it is called the dihedral quandle, and is denoted by R_n.
- If G is a group and we take the binary operation $a * b = ba^{-1}b$, then we get the core quandle, denoted as $\mathrm{Core}(G)$. In particular, if G is additive abelian, then $\mathrm{Core}(G)$ is the Takasaki quandle.
- Let A be an additive abelian group and $t \in \mathrm{Aut}(A)$. Then the set A equipped with the binary operation $a * b = ta + (\mathrm{id}_A - t)b$ is a quandle called the Alexander quandle of A with respect to t.

A quandle or rack X is called trivial if $x * y = x$ for all $x, y \in X$. Obviously, a trivial rack is a trivial quandle. Unlike groups, a trivial quandle can contain arbitrary number of elements. We denote the n-element trivial quandle by T_n.

We will see that, unlike groups, the quandle ring structure of trivial quandles is quite interesting.

Notice that, the rack axioms are equivalent to saying that for each $x \in X$, the map $S_x : X \to X$ given by

$$S_x(y) = y * x$$

is an automorphism of X. Further, in case of quandles, the axiom $x * x = x$ is equivalent to saying that S_x fixes x for each $x \in X$. Such an automorphism is called an inner automorphism of X, and the group generated by all such automorphisms is denoted by $\mathrm{Inn}(X)$.

Let X be a quandle and R an associative ring (not necessarily with unity). Let $R[X]$ be the set of all formal finite R-linear combinations of elements of X, that is,

$$R[X] := \Big\{ \sum_i \alpha_i x_i \mid \alpha_i \in R,\ x_i \in X \Big\}.$$

Then $R[X]$ is an additive abelian group in the usual way. Define a multiplication in $R[X]$ by setting

$$\big(\sum_i \alpha_i x_i \big) \cdot \big(\sum_j \beta_j x_j \big) := \sum_{i,j} \alpha_i \beta_j (x_i \cdot x_j).$$

Clearly, the multiplication is distributive with respect to addition from both left and right, and $R[X]$ forms a ring, which we call the quandle ring of X with coefficients in the ring R.

Since X is non-associative, unless it is a trivial quandle, it follows that $R[X]$ is a non-associative ring, in general. Analogously, if X is a rack, then we obtain the rack ring $R[X]$ of X with coefficients in the ring R.

Analogous to group rings, we define the augmentation map

$$\varepsilon : R[X] \to R$$

by setting

$$\varepsilon\Big(\sum_i \alpha_i x_i\Big) = \sum_i \alpha_i.$$

Clearly, ε is a surjective ring homomorphism, and $\Delta_R(X) := \ker(\varepsilon)$ is a two-sided ideal of $R[X]$, called the augmentation ideal of $R[X]$. Thus, we have

$$R[X]/\Delta_R(X) \cong R$$

as rings. In the case $R = \mathbb{Z}$, we denote the augmentation ideal simply by $\Delta(X)$.

Proposition [B.-Passi-Singh, 2019]

Let X be a rack and R an associative ring. Then $\{x - y \mid x, y \in X\}$ is a generating set for $\Delta_R(X)$ as an R-module. Further, if $x_0 \in X$ is a fixed element, then the set $\{x - x_0 \mid x \in X \setminus \{x_0\}\}$ is a basis for $\Delta_R(X)$ as an R-module.

Given a subrack Y of a rack X, it is natural to look for conditions under which $\Delta_R(Y)$ is a two-sided ideal of $R[X]$. For trivial racks, we have the following result.

Proposition [B.-Passi-Singh, 2019]

Let X be a trivial rack, Y a subrack of X and R an associative ring. Then $\Delta_R(Y)$ is a two-sided ideal of $R[X]$.

The next result characterises trivial quandles in terms of their augmentation ideals.

Theorem [B.-Passi-Singh, 2019]

Let X be a quandle and R an associative ring. Then the quandle X is trivial if and only if $\Delta_R^2(X) = \{0\}$.

Let X be a quandle and R an associative ring. For each $x_0 \in X$ and each two sided ideal I of $R[X]$, we define

$$X_{I,x_0} = \{x \in X \mid x - x_0 \in I\}.$$

Notice that, if $I = \Delta_R(X)$, then $X_{I,x_0} = X$, and if $I = \{0\}$, then $X_{I,x_0} = \{x_0\}$. In general, we have the following.

Theorem [B.-Passi-Singh, 2019]

Let X be a finite quandle and R an associative ring. Then for each $x_0 \in X$ and a two sided ideal I of $R[X]$, the set X_{I,x_0} is a subquandle of X. Further, there is a subset $\{x_1, \dots, x_m\}$ of X such that X is the disjoint union

$$X = X_{I,x_1} \sqcup \cdots \sqcup X_{I,x_m}.$$

Next, we proceed in the reverse direction of associating an ideal of $R[X]$ to a subquandle of X. Let $f : X \to Z$ be a quandle homomorphism. Consider the equivalence relation $\sim$ on X given by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Let $X/_\sim$ be the set of equivalence classes, where equivalence class of an element x is denoted by

$$X_x := \{x' \in X \mid f(x') = f(x)\}.$$

It is not difficult to prove that X_x is a subquandle of X for each $x \in X$.

The following is a sort of first isomorphism theorem for quandles.

Theorem [B.-Passi-Singh, 2019]

The binary operation given by $X_{x_1} \circ X_{x_2} = X_{x_1 \cdot x_2}$ gives a quandle structure on $X/_\sim$. Further, if $f : X \to Z$ is a surjective quandle homomorphism, then $X/_\sim \cong Z$ as quandles.

We say that a subquandle Y of a quandle X is normal if $Y = X_{x_0}$ for some $x_0 \in X$ and some quandle homomorphism $f : X \to Z$. In this case, we say that Y is normal based at x_0.
A pointed quandle, denoted (X, x_0), is a quandle X together with a fixed base point x_0. Let $f : (X, x_0) \to (Z, z_0)$ be a homomorphism of pointed quandles, and $Y = X_{x_0}$ a normal subquandle based at x_0. In this situation, we consider Y as the base point of $X/_\sim$, and denote $X/_\sim$ by X/Y. Then the natural map $x \mapsto X_x$ is a surjective homomorphism of pointed quandles

$$(X, x_0) \to (X/Y, X_{x_0}).$$

This further extends to a surjective ring homomorphism, say,

$$\pi : R[X] \to R[X/Y]$$

with $\ker(\pi)$ being a two sided ideal of $R[X]$.

Let (X, x_0) be a pointed quandle, $\mathcal{I}$ the set of two sided ideals of $R[X]$ and $\mathcal{S}$ the set of normal subquandles of X based x_0. Then there exist maps $\Phi : \mathcal{I} \to \mathcal{S}$ given by

$$\Phi(I) = X_{I,x_0}$$

and $\Psi : \mathcal{S} \to \mathcal{I}$ given by

$$\Psi(Y) = \ker(\pi).$$

With this set up, we have the following.

Theorem [B.-Passi-Singh, 2019]

Let (X, x_0) be a pointed quandle and R an associative ring. Then $\Phi\Psi = \mathrm{id}_{\mathcal{S}}$ and $\Psi\Phi \neq \mathrm{id}_{\mathcal{I}}$.

Assume that R is an associative ring with unity 1. Let X be a rack. Since $R[X]$ is a ring without unity, it is desirable to embed $R[X]$ into a ring with unity. The ring

$$R^\circ[X] = R[X] \oplus Re,$$

where e is a symbol satisfying $e\big(\sum_i \alpha_i x_i\big) = \sum_i \alpha_i x_i = \big(\sum_i \alpha_i x_i\big)e$, is called the extended rack ring of X.

We extend the augmentation map $\varepsilon : R^\circ[X] \to R$ to obtain the extended augmentation ideal

$$\Delta_{R^\circ}(X) := \ker(\varepsilon : R^\circ[X] \to R).$$

In the case $R = \mathbb{Z}$, we simply denote it by $\Delta_\circ(X)$. It is easy to see that the set $\{x - e \mid x \in X\}$ is a basis for $\Delta_{R^\circ}(X)$ as an R-module.

Proposition [B.-Passi-Singh, 2019]

(1) If X is a rack and $x_0 \in X$ a fixed element, then $\Delta_{R^\circ}(X) = \Delta_R(X) + R(e - x_0)$.

(2) If X is a quandle, then $\Delta^2_{R^\circ}(X) = \Delta_{R^\circ}(X)$.

Let X be a rack and R an associative ring with unity 1. Though the ring $R^\circ[X]$ has unity, it is non-associative, in general.

Problem

Determine maximal multiplicative subgroups of $R^\circ[X]$.

Let $\mathcal{U}(R^\circ[X])$ denote a maximal multiplicative subgroup of the ring $R^\circ[X]$. Notice that, $\varepsilon : R^\circ[X] \to R$ maps $\mathcal{U}(R^\circ[X])$ onto R^*, the group of units of R. Let

$$\mathcal{U}_1(R^\circ[X]) := \big\{r \in \mathcal{U}(R^\circ[X]) \mid \varepsilon(r) = 1\big\},$$

be the subgroup of normalized units. Then $\mathcal{U}(R^\circ[X]) = R^* \, \mathcal{U}_1(R^\circ[X])$, and one only need to compute the group of normalized units.

Define

$$\mathcal{V}(R^\circ[X]) := \{e + a \in \mathcal{U}(R^\circ[X]) \mid a \in R[X]\}.$$

Then $\mathcal{V}(R^\circ[X])$ is a normal subgroup of $\mathcal{U}(R^\circ[X])$ and

$$\mathcal{U}(R^\circ[X]) = R^*\mathcal{V}(R^\circ[X]).$$

To understand $\mathcal{V}(R^\circ[X])$ further, we define

$$\mathcal{V}_1(R^\circ[X]) := \mathcal{U}_1(R^\circ[X]) \cap \mathcal{V}(R^\circ[X]).$$

Theorem [B.-Passi-Singh, 2019]

Let X be a rack and R an associative ring with unity. Then $\mathcal{V}_1(R^\circ[X]) = \{e + a \in \mathcal{U}(R^\circ[X]) \mid a \in \Delta_R(X)\}$ and is a normal subgroup of $\mathcal{U}(R^\circ[X])$.

Let X be a rack and $x_0 \in X$ a fixed element. Define the set

$$\mathcal{V}_2(R^\circ[X]) = \{e + (\lambda - 1)x_0 \mid \lambda \in R^*\}.$$

Proposition [B.-Passi-Singh, 2019]

Let X be a rack, $x_0 \in X$ a fixed element and R an associative ring with unity. Then $\mathcal{V}_2(R^\circ[X])$ is a subgroup of $\mathcal{V}(R^\circ[X])$ and is isomorphic to R^*.

Theorem [B.-Passi-Singh, 2019]

Let X be a rack, $x_0 \in X$ a fixed element and R an associative ring with unity. Then there exists a split exact sequence

$$1 \longrightarrow \mathcal{V}_1(R^\circ[X]) \longrightarrow \mathcal{V}(R^\circ[X]) \longrightarrow \mathcal{V}_2(R^\circ[X]) \longrightarrow 1.$$

As a consequence, we have the following.

Corallary [B.-Passi-Singh, 2019]

Let X be a rack, $x_0 \in X$ and R an associative ring with unity. Then an arbitrary element $u = e + a_0 + (\lambda - 1)x_0 \in \mathcal{V}(R^\circ[X])$, $\lambda \in R^*$, is the product $u = u_1 u_2$, where $u_1 = e + a_0\big(e + (\lambda^{-1} - 1)x_0\big) \in \mathcal{V}_1(R^\circ[X])$ and $u_2 = e + (\lambda - 1)x_0 \in \mathcal{V}_2(R^\circ[X])$.

Since $\mathcal{V}_2(R^\circ[X]) \cong R^*$, it follows from Theorem that if we know the structure of $\mathcal{V}_1(R^\circ[X])$, then we can determine the structure of $\mathcal{V}(R^\circ[X])$.

If T is a trivial rack and R an associative ring with unity, then the ring $R^{\circ}[\mathrm{T}]$ is associative with unity, and hence $\mathcal{U}(R^{\circ}[\mathrm{T}])$ is precisely the group of units of $R^{\circ}[\mathrm{T}]$. As we observed, the main problem in determining $\mathcal{U}(R^{\circ}[\mathrm{T}])$ is the description of $\mathcal{V}_1(R^{\circ}[\mathrm{T}])$.

Proposition [B.-Passi-Singh, 2019]

Let T be a trivial rack and R an associative ring with unity. Then

$$\mathcal{V}_1(R^{\circ}[\mathrm{T}]) = \{e + a \mid a \in \Delta_R(\mathrm{T})\}$$

is an abelian subgroup of $\mathcal{U}_1(R^{\circ}[\mathrm{T}])$. Further, $\mathcal{V}_1(R^{\circ}[\mathrm{T}]) \cong \Delta_R(\mathrm{T})$.

More generally, we prove the following.

Theorem [B.-Passi-Singh, 2019]

Let T be a trivial rack, $x_0 \in \mathrm{T}$ and R an associative ring with unity. Then $\mathcal{U}_1(R^\circ[\mathrm{T}]) = \left\{ e + a + \alpha(x_0 - e) \mid a \in \Delta_R(\mathrm{T}) \text{ and } \alpha - 1 \in R^* \right\}$.

As a consequence, we have the following for integral coefficients.

Corollary

Let T be a trivial rack, $x_0 \in X$ and R an associative ring with unity. Then the following statements hold:

1. $\mathcal{U}(\mathbb{Z}^\circ[\mathrm{T}]) = \pm\big\{e + a + \alpha(x_0 - e) \mid a \in \Delta_R(\mathrm{T}) \text{ and } \alpha = 0, 2\big\}$.
2. If $\mathrm{T}_1 = \{x_0\}$, then $\mathcal{U}(\mathbb{Z}^\circ[\mathrm{T}_1]) = \{\pm e, 2x - e, -2x + e\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Next we consider nilpotency of $\mathcal{V}(R^\circ[\mathrm{T}])$. Obviously, $\mathcal{V}(R^\circ[\mathrm{T}_1])$ is nilpotent. In the general case, we prove the following.

Proposition [B.-Passi-Singh, 2019]

Let R be an associative ring with unity such that $|R^*| > 1$. If T is a trivial rack with more than one element, then $\mathcal{V}(R^\circ[\mathrm{T}])$ is not nilpotent.

Let X be a rack and R an associative ring. Consider the direct sum

$$\mathcal{X}_R(X) := \sum_{i \geq 0} \Delta_R^i(X)/\Delta_R^{i+1}(X)$$

of R-modules $\Delta_R^i(X)/\Delta_R^{i+1}(X)$. We regard $\mathcal{X}_R(X)$ as a graded R-module with the convention that the elements of $\Delta_R^i(X)/\Delta_R^{i+1}(X)$ are homogeneous of degree i.

Defining multiplication in $\mathcal{X}_R(X)$, we see that $\mathcal{X}_R(X)$ becomes a graded ring, and we call it the associated graded ring of $R[X]$.

For trivial quandles we have a full description of its associated graded ring.

Proposition [B.-Passi-Singh, 2019]

If T is a trivial quandle and $x_0 \in \mathrm{T}$, then $\mathcal{X}_R(\mathrm{T}) = Rx_0 \oplus \Delta_R(\mathrm{T})$.

For studying $\mathcal{X}_R(X)$ we need to understand the quotients

$$\Delta_R^i(X)/\Delta_R^{i+1}(X).$$

In the case of groups, we have the following result: Let G be a finite group and $Q_n(G) = \Delta_{\mathbb{Z}}^n(G)/\Delta_{\mathbb{Z}}^{n+1}(G)$, then there exist integers n_0 and π such that

$$Q_n(G) \cong Q_{n+\pi}(G) \text{ for all } n \geq n_0.$$

We compute powers of the integral augmentation ideals of the dihedral quandles R_n for some small values of n.

Observe that $\mathrm{R}_2 = \mathrm{T}_2$, the trivial quandle with 2 elements.

Consider the integral quandle ring of the dihedral quandle $\mathrm{R}_3 = \{a_0, a_1, a_2\}$. Set $e_1 := a_1 - a_0$ and $e_2 := a_2 - a_0$. Then $\Delta(\mathrm{R}_3) = \langle e_1, e_2 \rangle$, $\Delta^2(\mathrm{R}_3) = \langle e_1 + e_2, 3e_2 \rangle$ and $\Delta(\mathrm{R}_3)/\Delta^2(\mathrm{R}_3) \cong \mathbb{Z}_3$.

Proposition [B.-Passi-Singh, 2019]

The following holds for each natural number k:

$$\Delta^{2k-1}(\mathrm{R}_3) = \langle 3^{k-1}e_1, 3^{k-1}e_2 \rangle, \quad \Delta^{2k}(\mathrm{R}_3) = \langle 3^{k-1}(e_1 + e_2), 3^k e_2 \rangle,$$

$$\Delta^k(\mathrm{R}_3)/\Delta^{k+1}(\mathrm{R}_3) \cong \mathbb{Z}_3.$$

From the preceding proposition, we obtain the infinite filtration

$$\Delta(\mathrm{R}_3) \supseteq \Delta^2(\mathrm{R}_3) \supseteq \Delta^3(\mathrm{R}_3) \supseteq \dots$$

such that

$$\cap_{n \geq 0} \Delta^n(\mathrm{R}_3) = \{0\}.$$

Hence, $\Delta^n(\mathrm{R}_3)$ in not nilpotent, but is residually nilpotent.

Next, we calculate the powers of augmentation ideal of R_4. First, notice that

$$\mathbb{Z}[\mathrm{R}_4] = \mathbb{Z}a_0 \oplus \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \oplus \mathbb{Z}a_3$$

and

$$\Delta(\mathrm{R}_4) = \mathbb{Z}(a_1 - a_0) \oplus \mathbb{Z}(a_2 - a_0) \oplus \mathbb{Z}(a_3 - a_0).$$

Let us set $e_1 := a_1 - a_0, \quad e_2 := a_2 - a_0, \quad e_3 := a_3 - a_0.$

Proposition [B.-Passi-Singh, 2019]

1. $\Delta^2(\mathrm{R}_4)$ is generated as an abelian group by the set $\{e_1 - e_2 - e_3, 2e_2\}$ and $\Delta(\mathrm{R}_4)/\Delta^2(\mathrm{R}_4) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.
2. If $k > 2$, then $\Delta^k(\mathrm{R}_4)$ is generated as an abelian group by the set $\{2^{k-1}(e_1 - e_2 - e_3), 2^k e_2\}$ and $\Delta^{k-1}(\mathrm{R}_4)/\Delta^k(\mathrm{R}_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Problem

We suppose that for be the dihedral quandle R_n *and a ring* R *hold:*

1. *If* $n > 1$ *is an odd integer, then* $\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n) \cong \mathbb{Z}_n$ *for all* $k \geq 1$.
2. *If* $n > 2$ *is an even integer, then* $|\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n)| = n$ *for all* $k \geq 2$.

The following result gives positive answer on our first question.

Theorem [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let n be odd. Then $\Delta^k(\mathrm{R}_n)/\Delta^{k+1}(\mathrm{R}_n) \cong \mathbb{Z}_n$ for all $k \geq 1$.

The following result gives partial answer on the second question.

Theorem [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let $n = 2k$ for some positive integer k. Then $\Delta(\mathrm{R}_n)/\Delta^2(\mathrm{R}_n) \cong \mathbb{Z} \oplus \mathbb{Z}_k$.

We know that any trivial quandle is associative, but an arbitrary quandle, and hence its quandle ring need not be associative. In particular, the dihedral quandle R_3, and hence all dihedral quandles R_n, $n \geq 3$ are not associative.

Recall that, a ring R is called power-associative if every element of R generates an associative subring of R. If T is a trivial quandle and R an associative ring, then $R[\mathrm{T}]$ is associative, and hence power-associative.

In general, it is an interesting question to determine the conditions under which the ring $R[X]$ is power-associative.

We investigate power-associativity of quandle rings of dihedral quandles R_n. The cases $n = 1, 2$ are obvious. For $n = 3$, we prove the following result.

Proposition [B.-Passi-Singh, 2019]

Let R be a commutative ring with unity of characteristic not equal to 2, 3 or 5. Then the quandle ring $R[\mathrm{R}_3]$ is not power-associative.

Proposition [B.-Passi-Singh, 2019]

Let R be a commutative ring with unity of characteristic not equal to 2. Then $R[\mathrm{R}_n]$ is not power-associative for $n > 3$.

Theorem [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let R be an associative ring with unity of characteristic not equal to 2 and 3. Then the quandle ring $R[X]$ is not power-associative if X is a non-trivial quandle.

Problem

1. *Let X and Y be two racks such that $R[X] \cong R[Y]$. Does it follow that $X \cong Y$?*
2. *Let X and Y be two racks with $\mathcal{X}_R(X) \cong \mathcal{X}_R(Y)$. Does it follow that $X \cong Y$?*

Proposition [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let X a quandle of order 3. Then the three quandle rings arising from X are not pairwise isomorphic.

Proposition [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let R a field of characteristic 3. Then thee are two non-isomorphic quandles X and Y of cardinality 4 such that their quandle rings $R[X]$ and $R[X]$ are isomorphic.

Proposition [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

Let R a field of characteristic 0. Then thee are two non-isomorphic quandles X and Y of cardinality 7 such that their quandle rings $R[X]$ and $R[X]$ are isomorphic.

The following result gives negative answer on our second question.

Proposition [M. Elhamdadi - N. Fernando - B. Tsvelikovskiy, 2019]

There are two non-isomorphic quandles X and Y of cardinality 4 such that $\mathcal{X}_R(X) \cong \mathcal{X}_R(Y)$.

It is easy to see that the zero $0 \in R$ lies in $R[X]$ and is a trivial one-element quandle, which we will called zero quandle. On the other side if some quandle $Q \subseteq R[X]$ containes 0, then $Q = \{0\}$ is the zero quandle. We will denote $mq(R[X])$ the set of non-zero maximal quadles in $R[X]$. If $mq(R[X]) = \{X\}$ we will say that $R[X]$ is quandle ring with unique maximal quandle. We formulate

Problem

For a ring R and a quadle X find the set $mq(R[X])$.

The first step to solution of this problem is a description of idempotents in $R[X]$, i.e. elements z in $R[X]$ such that $z^2 = z$. Let us denote $I(R[X])$ the set of all non-zero idempotents in $R[X]$.

Problem

For a ring R and a quadle X find the set of all non-zero idempotents $I(R[X])$.

The problem of description the set $mq(R[X])$ connects with description of the automorphism group $\mathrm{Aut}(R[X])$, that is the group of automorphisms of $R[X]$, which fix R.

Problem

Let X be a (finite) quandle. Find $\mathrm{Aut}(R[X])$. What is connection between this group and $\mathrm{Aut}(X)$?

R. Zh. Aleev for group rings formulated the following problem (see Kourovka Notebook, Problem 13.1):

Problem

Let $U(R)$ denote the group of units of a ring R. Let G be a finite group, $\mathbb{Z}[G]$ the integral group ring of G, and $\mathbb{Z}_p[G]$ the group ring of G over the residues modulo a prime number p.

Describe the homomorphism from $U(\mathbb{Z}[G])$ into $U(\mathbb{Z}_p[G])$ induced by reducing the coefficients modulo p. More precisely, find the kernel and the image of this homomorphism and an explicit transversal over the kernel.

For the maximal quandles we can formulate the similar problem

Problem

Let Q be a finite quandle. Describe the map from $mq(\mathbb{Z}[Q])$ into $mq(\mathbb{Z}_p[Q])$ induced by reducing the coefficients modulo p.

If z is is an idempotent of $R[X]$, then $\varepsilon(z) = \varepsilon(z)^2$. Hence, $\varepsilon(z) = 0$ or $\varepsilon(z) = 1$. In the first case $z \in \Delta_R(X)$, in the second case z is presented in the form $z = x + \delta$ for some $x \in X$ and $\delta \in \Delta_R(X)$.

Proposition [B.-Passi-Singh, 2019]

If a quandle $X = X_1 \sqcup X_2$ is a disjoint union of two subquandles, then

$$I(R[X]) \supseteq \left(I(R[X_1]) \cup I(R[X_2])\right).$$

Considering the 3-element quandle $Cz(4)$, we see that the inclusion in this proposition is strict.

Let $\mathrm{T}_n = \{x_1, x_2, \ldots, x_n\}$ be the n-element trivial quandle. It is not difficult to prove

Proposition [B.-Passi-Singh, 2019]

The set of non-zero idempotents in $\mathbb{Z}[\mathrm{T}_n]$ has the form

$$I(\mathbb{Z}[\mathrm{T}_n]) = x_1 + \Delta(\mathrm{T}_n).$$

Consider the 3-element dihedral quandle $\mathrm{R}_3 = \{a_0, a_1, a_2\}$.

Proposition [B.-Passi-Singh, 2019]

$$I(\mathbb{Z}[\mathrm{R}_3]) = \{a_0, a_1, a_2\}.$$

Take the dihedral quandle $\mathrm{R}_4 = \{a_0, a_1, a_2, a_3\}$ with four elements. R_4 is not connected since the elements a_0 and a_1 lying in different orbits. In fact R_4 is the disjoint union of two trivial subquandles:
$\mathrm{R}_4 = \{a_0, a_2\} \sqcup \{a_0, a_2\}$. Hence,

$$I(\mathbb{Z}[\mathrm{R}_4]) \supseteq \left(I(\mathbb{Z}[\{a_0, a_2\}]) \cup I(\mathbb{Z}[\{a_1, a_3\}])\right).$$

In fact we have equality here, i.e. the following proposition holds

Proposition [B.-Passi-Singh, 2019]

$$I(\mathbb{Z}[\mathrm{R}_4]) = \{a_0 + \alpha(a_2 - a_0), \;\; a_1 + \beta(a_3 - a_1) \mid\mid \alpha, \beta \in \mathbb{Z}\}.$$

At first find maximal quandles in the quandle ring $\mathbb{Z}[\mathrm{T}_n]$.

Theorem [B.-Passi-Singh, 2019]

The maximal non-zero quandle in $\mathbb{Z}[\mathrm{T}_n]$ is unique trivial quandl which consists of elements

$$x_1 + \Delta(\mathrm{T}_n).$$

As we seen the quandle ring $\mathbb{Z}[\mathrm{R}_3]$ has unique maximal quandle which is R_3, i.e. $mq(\mathbb{Z}[\mathrm{R}_3]) = \{\mathrm{R}_3\}$. The quandle R_3 is connected. For connected quandles we can formulate

Problem

Let X be a (finite) connected quandle. Is it true that $mq(\mathbb{Z}[X]) = \{X\}$?

For non-connected quandles it is not true.

Theorem [B.-Passi-Singh, 2019]

The maximal non-zero quandle in $\mathbb{Z}[\mathrm{R}_4]$ is unique and consists of elements

$$M = \{a_0 + \alpha(a_2 - a_0), \quad a_1 + \beta(a_3 - a_1) \mid\mid \alpha, \beta \in \mathbb{Z}\}.$$

We proved that $mq(\mathbb{Z}[\mathrm{R}_3]) = \{\mathrm{R}_3\}$. Consider homomorphism $\varphi_2 : \mathbb{Z}[\mathrm{R}_3] \to \mathbb{Z}_2[\mathrm{R}_3]$ and find $mq(\mathbb{Z}_2[\mathrm{R}_3])$. The quandle ring $\mathbb{Z}_2[\mathrm{R}_3]$ contains 8 elements and it is easy to see that all its elements are idempotents.

Proposition [B.-Passi-Singh, 2019]

The set $mq(\mathbb{Z}_2[\mathrm{R}_3])$ contains three quandles: 1-element quandle $\{a_0 + a_1 + a_2\}$ and two isomorphic 3-elements quandles: R_3 and $\{a_0 + a_1, a_0 + a_2, a_1 + a_2\}$.

Corollary

The homomorphism $\varphi_2 : \mathbb{Z}[\mathrm{R}_3] \to \mathbb{Z}_2[\mathrm{R}_3]$ induces a map $mq(\mathbb{Z}[\mathrm{R}_3]) \to mq(\mathbb{Z}_2[\mathrm{R}_3])$ which is not surjective.

For a quandle ring $R[X]$ denote $\mathrm{Aut}(R[X])$ the group of automorphisms of $R[X]$ which fix elements of R. It is obvious that $\mathrm{Aut}(X) \leq \mathrm{Aut}(R[X])$. On the other side, if $|X| = n$ is a finite quandle, then $\mathrm{Aut}(R[X]) \leq \mathrm{GL}_n(R)$. If $\varphi \in \mathrm{Aut}(R[X])$, then it is defined by the actions on the elements of X. Suppose that $X = \{x_1, x_2, \ldots, x_n\}$. Then

$$\varphi(x_i) = \sum_j \alpha_{ij} x_j, \quad \alpha_{ij} \in R,$$

are idempotents for all elements $x_i \in X$ and the quandle $\varphi(X)$ isomorphic to X.

Consider the group $\mathrm{Aut}(\mathbb{Z}[\mathrm{T}_n])$, where $\mathrm{T}_n = \{x_1, x_2, \ldots, x_n\}$ is the n-element trivial quandle. We know that $\mathrm{Aut}(\mathrm{T}_n)$ is isomorphic to the symmetric group Σ_n.

It is easy to see that for $n = 1$ the group $\mathrm{Aut}(\mathbb{Z}[\mathrm{T}_1])$ is trivial. Then we will assume that $n > 1$. If $\varphi \in \mathrm{Aut}(\mathbb{Z}[\mathrm{T}_n])$, then $\varphi(\mathrm{T}_n)$ is a n-element trivial quandle and φ is an automorphism of $\mathbb{Z}$-module $\mathbb{Z}[\mathrm{T}_n]$.

In the case $n = 2$ we have

Proposition [B.-Passi-Singh, 2019]

$$\mathrm{Aut}(\mathbb{Z}[\mathrm{T}_2]) \cong \mathbb{Z} \rtimes \Sigma_2.$$

Problem

Find $\mathrm{Aut}(\mathbb{Z}[\mathrm{T}_n])$ *for* $n > 2$*.*

For the quandle R_3 we know that the ring $\mathbb{Z}[\mathrm{R}_3]$ contains unique maximal quandle that is isomorphic to R_3. Hence

Proposition [B.-Passi-Singh, 2019]

$\mathrm{Aut}(\mathbb{Z}[\mathrm{R}_3]) \cong \mathrm{Aut}(\mathrm{R}_3)$.

Problem

Let X *be a (finite) connected quandle. Is it true that* $\mathrm{Aut}(\mathbb{Z}[X]) = \mathrm{Aut}(X)$*?*

Thank you!