THE DRESSING CHAIN AND ONE–POINT COMMUTING DIFFERENCE OPERATORS OF RANK 1.

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We denote by $\tilde{L}_k, \ \tilde{L}_s$ the operators of orders $k = N_- + N_+$ and $s = M_- + M_+$

$$\tilde{L}_k = \sum_{j=-N_-}^{N_+} u_j(n) T^j, \qquad \tilde{L}_s = \sum_{j=-M_-}^{M_+} v_j(n) T^j,$$

where $n \in \mathbb{Z}, \ N_\pm, M_\pm \geq 0$, T is the shift operator

$$Tf(n) = f(n+1), \qquad f : \mathbb{Z} \to \mathbb{C}.$$

If two difference operators $\tilde{L}_k$ and $\tilde{L}_s$ commute, then there is a nonzero polynomial $F(z, w)$ such that $F(\tilde{L}_k, \tilde{L}_s) = 0$. The polynomial F defines the *spectral curve* of the pair $\tilde{L}_k,\ \tilde{L}_s$

$$\Gamma = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\}.$$

The common eigenvalues are parametrized by the spectral curve

$$\tilde{L}_k\psi = z\psi, \quad \tilde{L}_s\psi = w\psi, \quad (z,w) \in \Gamma.$$

The dimension of the space of common eigenfunctions of the pair $\tilde{L}_k,\ \tilde{L}_s$ for fixed eigenvalues is called the *rank* of $\tilde{L}_k,\ \tilde{L}_s$

$$l = \dim\{\psi : \tilde{L}_k\psi = z\psi, \quad \tilde{L}_s\psi = w\psi, \quad (z,w) \in \Gamma.\}$$

Any commutative ring of difference operators in one discrete variable is isomorphic to the ring of meromorphic functions on a spectral curve with s fixed poles (I. M. Krichever, S. P. Novikov). Such operators are said to be s–point.

Spectral data for two–point operators of rank 1 were found by I. M. Krichever and examples of such operators also were found by D. Mumford. Eigenfunctions for two–point operators of rank 1 (Baker–Akhiezer functions) can be found explicitly in terms of theta function of the spectral curves. Spectral data for one–point operators of rank $l > 1$ were obtained by I. M. Krichever and S. P Novikov. These operators play an important role in constructing algebro–geometric solutions of $1D$ and $2D$ Toda chains. One–point Krichever–Novikov operators of rank 2 were studied by G. S. Mauleshova and A. E. Mironov; in particular, examples of such operators for hyperelliptic spectral curves of any genus were constructed.

Consider the hyperelliptic spectral curve Γ defined by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g} z^{2g} + \ldots + c_0,$$

for the base point we take $q = \infty$. Let $\psi(n, P)$ be the corresponding to the Baker–Akhiezer function. Then there exist commuting operators $\tilde{L}_2,\ \tilde{L}_{2g+1}$ such that

$$\tilde{L}_2 \psi = \big((T + U_n)^2 + W_n\big)\psi = z\psi, \quad \tilde{L}_{2g+1}\psi = w\psi.$$

Example 1

The operator

$$L_2^\sharp = (T + r_1\cos(n))^2 + \frac{r_1^2\sin(g)\sin(g+1)}{2\cos^2(g+\frac{1}{2})}\cos(2n),$$

$r_1 \neq 0$ commutes with a operator $L_{2g+1}^\sharp$.

Example 2

The operator

$$L_2^\checkmark = (T + \alpha_2 n^2 + \alpha_0)^2 - g(g+1)\alpha_2^2 n^2, \quad \alpha_2 \neq 0$$

commutes with a operator $L_{2g+1}^\checkmark$.

We consider one–point ε–difference operators of rank 1 having the form

$$L_k = \frac{T_\varepsilon^k}{\varepsilon^k} + u_{k-1}(x,\varepsilon)\frac{T_\varepsilon^{k-1}}{\varepsilon^{k-1}} + \ldots + u_0(x,\varepsilon),$$

where T_ε is the operator of shift by ε, i.e., $T_\varepsilon f(x) = f(x+\varepsilon)$, $f : \mathbb{C} \to \mathbb{C}$. Let Γ be the hyperelliptic spectral curve determined by the equation

$$w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0,$$

and let $q = \infty$. Suppose that the operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + B(x,\varepsilon)$$

commutes with L_{2g+1}.

Consider the function $A_g(x, \varepsilon)$ defined as follows. We put

$$A_1 = -2\zeta(\varepsilon) - \zeta(x - \varepsilon) + \zeta(x + \varepsilon)$$

and

$$A_2 = -\frac{3}{2}\big(\zeta(\varepsilon) + \zeta(3\varepsilon) + \zeta(x - 2\varepsilon) - \zeta(x + 2\varepsilon)\big),$$

where $\zeta(x)$ is the Weierstrass function. Next, for odd $g = 2g_1 + 1$, we put

$$A_g = A_1 \prod_{k=1}^{g_1} \left(1 + \frac{\zeta(x - (2k+1)\varepsilon) - \zeta(x + (2k+1)\varepsilon)}{\zeta(\varepsilon) + \zeta((4k+1)\varepsilon)}\right),$$

and for even $g = 2g_1$, we put

$$A_g = A_2 \prod_{k=2}^{g_1} \left(1 + \frac{\zeta(x - 2k\varepsilon) - \zeta(x + 2k\varepsilon)}{\zeta(\varepsilon) + \zeta((4k-1)\varepsilon)}\right).$$

Example 3

The operator

$$L_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + A_g(x,\varepsilon)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon)$$

commutes with L_{2g+1}. Moreover,

$$L_2 = \partial_x^2 - g(g+1)\wp(x) + O(\varepsilon).$$

Let

$$\hat{L}_2 = \frac{T_\varepsilon^2}{\varepsilon^2} + (u(x,t,\varepsilon) + u(x+\varepsilon,t,\varepsilon))\frac{T_\varepsilon}{\varepsilon} - v(x,\varepsilon).$$

We consider the one–point algebraic–geometric solution of rank one

$$\partial_t u(x,t,\varepsilon) + \partial_t u(x+\varepsilon,t,\varepsilon) = \tag{1}$$

$$u^2(x,t,\varepsilon) - u^2(x+\varepsilon,t,\varepsilon) + v(x,\varepsilon) - v(x+\varepsilon,\varepsilon).$$

Equation (3) is equivalent to the commutativity condition

$$[\hat{L}_2, \partial_t - (\frac{T_\varepsilon}{\varepsilon} + u(x,t,\varepsilon))] = 0.$$

Theorem 1

For $g = 1$, the one–point algebraic-geometric solution of rank one of equation (1) has the form

$$v(x,\varepsilon) = \gamma(x,\varepsilon)+\gamma(x+\varepsilon,\varepsilon)-\left(\frac{\sqrt{F_1(\gamma(x,\varepsilon))}+\sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\gamma(x,\varepsilon)-\gamma(x+\varepsilon,\varepsilon)}\right)^2,$$

$$u(x,t,\varepsilon) = -\frac{\sqrt{F_1(\gamma(x,\varepsilon))}+\sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\gamma(x,\varepsilon)-\gamma(x+\varepsilon,\varepsilon)}-$$

$$\frac{\sqrt{F_1(\wp(t))}+\sqrt{F_1(\gamma(x,\varepsilon))}}{\wp(t)-\gamma(x,\varepsilon)}+\frac{\sqrt{F_1(\wp(t))}+\sqrt{F_1(\gamma(x+\varepsilon,\varepsilon))}}{\wp(t)-\gamma(x+\varepsilon,\varepsilon)},$$

where $F_1(z) = z^3 + c_1 z + c_0$, $\gamma(x,\varepsilon)$ is any function parameter, $\wp(t)$ is the Weierstrass elliptic function satisfying the equation

$$(\wp'(t))^2 = 4F_1(\wp(t)). \qquad (*)$$

The operators $\hat{L}_2,\ \hat{L}_3$ satisfy the equation $\hat{L}_3^2 = F_1(\hat{L}_2)$.

If

$$\gamma(x,\varepsilon) = \wp(x-\varepsilon),$$

then

$$\hat{L}_2 = \frac{T_\varepsilon^2}{\varepsilon^2} - \big(2\zeta(\varepsilon) + \zeta(x-\varepsilon+t) - \zeta(x+\varepsilon+t)\big)\frac{T_\varepsilon}{\varepsilon} + \wp(\varepsilon),$$

$$\hat{L}_3 = \frac{T_\varepsilon^3}{\varepsilon^3} - \big(3\zeta(\varepsilon) + \zeta(x-\varepsilon+t) - \zeta(x+2\varepsilon+t)\big)\frac{T_\varepsilon^2}{\varepsilon^2} +$$

$$\big((\zeta(\varepsilon) + \zeta(x-\varepsilon+t) - \zeta(x+t))(\zeta(\varepsilon) + \zeta(x+t) - \zeta(x+\varepsilon+t)) +$$

$$2\wp(\varepsilon) + \wp(x+t)\big)\frac{T_\varepsilon}{\varepsilon} + \frac{1}{2}\wp'(\varepsilon),$$

$$\partial_t - \big(\frac{T_\varepsilon}{\varepsilon} + u(x,t,\varepsilon)\big) = \partial_t - \big(\frac{T_\varepsilon}{\varepsilon} - \zeta(\varepsilon) - \zeta(x-\varepsilon+t) + \zeta(x+t)\big),$$

where $\zeta(z)$ is the Weierstrass elliptic function.

Moreover,

$$\hat{L}_2 = \big(\partial_x^2 - 2\wp(x+t)\big) + O(\varepsilon),$$

$$\hat{L}_3 = \big(\partial_x^3 - 3\wp(x+t)\partial_x - \frac{3}{2}\wp'(x+t)\big) + O(\varepsilon),$$

$$\partial_t - \big(\frac{T_\varepsilon}{\varepsilon} - \zeta(\varepsilon) - \zeta(x-\varepsilon+t) + \zeta(x+t)\big) = (\partial_t - \partial_x) + O(\varepsilon).$$

Herewith, the spectral curve of the pair of commuting differential operators

$$\partial_x^2 - 2\wp(x+t), \qquad \partial_x^3 - 3\wp(x+t)\partial_x - \frac{3}{2}\wp'(x+t)$$

is the same as for ε–difference operators $\hat{L}_2,\ \hat{L}_3$.