Simplicial structure on the groups of virtual pure braids

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A sequence of sets $\mathcal{X} = \{X_n\}_{n \geq 0}$ is called a simplicial set if there are face maps:

$$d_i : X_n \longrightarrow X_{n-1} \text{ for } 0 \leq i \leq n$$

and degeneracy maps

$$s_i : X_n \longrightarrow X_{n+1} \text{ for } 0 \leq i \leq n.$$

This maps satisfy the following simplicial identities:

$$\begin{array}{lll} d_i d_j = d_{j-1} d_i & \text{if} & i < j, \\ s_i s_j = s_{j+1} s_i & \text{if} & i \leq j, \\ d_i s_j = s_{j-1} d_i & \text{if} & i < j, \\ d_j s_j = id = d_{j+1} s_j, & & \\ d_i s_j = s_j d_{i-1} & \text{if} & i > j+1. \end{array}$$

A simplicial group $\mathcal{G} = \{G_n\}_{n\geq 0}$ consists of a simplicial set $\mathcal{G}$ for which each G_n is a group and each d_i and s_i is a group homomorphism.

Examples:

1) Simplicial circle S^1_*: Let $S^1 = \Delta[1]/\partial\Delta[1]$ be a circle. Define

$$S^1_0 = \{*\},\ \ S^1_1 = \{*, \sigma\},\ \ S^1_2 = \{*, s_0\sigma, s_1\sigma\}, \ldots, S^1_n = \{*, x_0, \ldots, x_{n-1}\}, \ldots$$

where $x_i = s_{n-1} \ldots \widehat{s}_i \ldots s_0\sigma$. It is not difficult to check that S^1_* is a simplicial set.

2) Free simplicial group F_*: Let $F_0 = \{e\}$ be the trivial group, $F_1 = \langle y \rangle$ be the infinite cyclic group, $F_2 = \langle s_0y, s_1y \rangle$ be the free group of rank 2, $F_n = \langle y_0, \ldots, y_{n-1}\}$, where $y_i = s_{n-1} \ldots \widehat{s}_i \ldots s_0y$. It is not difficult to check that F_* is a simplicial group.

Milnor's $F[S^1]$-construction gives a possibility to define the homotopy groups $\pi_n(S^2)$ combinatorially, in terms of free groups. The $F[S^1]$-construction is a free simplicial group with the following terms

$$
\begin{aligned}
&F[S^1]_0 = 1,\\
&F[S^1]_1 = F(\sigma),\\
&F[S^1]_2 = F(s_0\sigma, s_1\sigma),\\
&F[S^1]_3 = F(s_i s_j \sigma \mid 0 \leq j \leq i \leq 2),\\
&\ldots
\end{aligned}
$$

The face and degeneracy maps are determined with respect to the standard simplicial identities for these simplicial groups.

Milnor proved that the geometric realization of $F[S^1]$ is weakly homotopically equivalent to the loop space $\Omega S^2 = \Omega\Sigma S^1$. Hence, the homotopy groups of the Moore complex of $F[S^1]$ are naturally isomorphic to the homotopy groups $\pi_n(S^2)$:

$$\pi_n(F[S^1]) = Z_n(F[S^1])/B_n(F[S^1]) \simeq \pi_{n+1}(S^2).$$

The Moore complex $N\mathcal{G} = \{N_n\mathcal{G}\}_{n\geq 0}$ of a simplicial group $\mathcal{G}$ is defined by

$$N_n\mathcal{G} = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i : G_n \longrightarrow G_{n-1}).$$

Then $d_0(N_n\mathcal{G}) \subseteq N_{n-1}\mathcal{G}$ and $N\mathcal{G}$ with d_0 is a chain complex of groups. An element in

$$\mathrm{B}_n\mathcal{G} = d_0(N_{n+1}\mathcal{G})$$

is called a Moore boundary and an element in

$$\mathrm{Z}_n\mathcal{G} = \mathrm{Ker}(d_0 : N_n\mathcal{G} \longrightarrow N_{n-1}\mathcal{G})$$

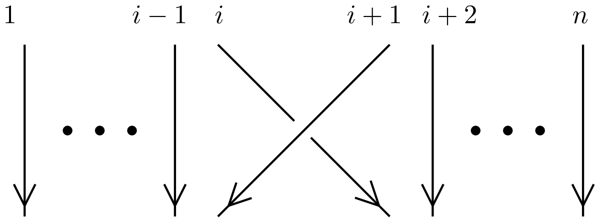
is called a Moore cycle. The nth homotopy group $\pi_n(\mathcal{G})$ is defined to be the group

$$\pi_n(\mathcal{G}) = H_n(N\mathcal{G}) = \mathrm{Z}_n\mathcal{G}/\mathrm{B}_n\mathcal{G}.$$

Braid group B_n on $n \geq 2$ strands is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and is defined by relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, 2, \ldots, n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i && \text{for } |i-j| \geq 2. \end{aligned}$$

The generators σ_i have the following geometric interpretation:



σ_i

There is a homomorphism $\varphi : B_n \longrightarrow S_n$, $\varphi(\sigma_i) = (i, i+1)$, $i = 1, 2, \ldots, n-1$. Its kernel $\mathrm{Ker}(\varphi)$ is called the pure braid group and is denoted by P_n. Note that P_2 is infinite cyclic group.

Markov proved that P_n is a semi-direct product of free groups:

$$P_n = U_n \leftthreetimes U_{n-1} \leftthreetimes \ldots \leftthreetimes U_2,$$

where $U_k \simeq F_{k-1}$, $k = 2, 3, \ldots, n$, is a free group of rank k.

F. Cohen and J. Wu (2011) defined simplicial group $AP_* = \{AP_n\}_{n\geq 0}$, where $AP_n = P_{n+1}$ with face and degeneracy maps corresponding to deleting and doubling of strands, respectively. They proved that AP_* is contractible (hence $\pi_n(AP_*)$ is trivial group for all n).

On the other side, F. Cohen and J. Wu constructed an injective canonical map of simplicial groups

$$\Theta : F[S^1] \longrightarrow AP_*,$$

This leads to the conclusion that the cokernel of Θ is homotopy equivalent to S^2. Hence, it is possible to present generators of $\pi_n(S^2)$ by pure braids.

Denote $c_{11} = \sigma_1^{-2} \in P_2$ and T_*^c be a simplicial subgroup of AP_* that is generated by c_{11}, i.e.

$$T_0 = 1, \quad T_1 = \langle c_{11} \rangle, \quad T_2 = \langle c_{21}, c_{12} \rangle, \quad T_3 = \langle c_{31}, c_{22}, c_{13} \rangle, \quad \ldots,$$

where

$$c_{21} = s_0 c_{11}, \;\; c_{12} = s_1 c_{11}, \;\; c_{31} = s_1 s_0 c_{11}, \;\; c_{22} = s_2 s_0 c_{11}, \;\; c_{13} = s_2 s_1 c_{11}, \ldots$$

Then $\Theta(F[S^1]) = T_*^c$.

It is not difficult to see that

$$P_n = \langle T_2, T_3, \ldots, T_{n-1} \rangle.$$

Hence, P_n is generated by elements that come from c_{11} with the cabling operations.

Question

What is a set of defining relations of P_n into the generators c_{ij}?

Proposition [V. B, R. Mikhailov, J. Wu, 2018]

The group P_4 is generated by elements

$$c_{11}, \quad c_{21}, \quad c_{12}, \quad c_{31}, \quad c_{22}, \quad c_{13}$$

and is defined by relations (where $\varepsilon = \pm 1$):

$$c_{21}^{c_{11}^{\varepsilon}} = c_{21}, \quad c_{12}^{c_{11}^{\varepsilon}} = c_{12}^{c_{21}^{-\varepsilon}}, \quad c_{31}^{c_{11}^{\varepsilon}} = c_{31}, \quad c_{22}^{c_{11}^{\varepsilon}} = c_{22}, \quad c_{13}^{c_{11}^{\varepsilon}} = c_{13}^{c_{22}^{-\varepsilon}},$$

$$c_{31}^{c_{21}^{\varepsilon}} = c_{31}, \quad c_{22}^{c_{21}^{\varepsilon}} = c_{22}^{c_{31}^{-\varepsilon}}, \quad c_{13}^{c_{21}^{\varepsilon}} = c_{13}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}},$$

$$c_{31}^{c_{12}^{\varepsilon}} = c_{31}, \quad c_{13}^{c_{12}^{\varepsilon}} = c_{13}^{c_{31}^{-\varepsilon}}.$$

$$c_{22}^{c_{12}^{-1}} = c_{13}^{c_{31}} c_{13}^{-c_{22}} c_{22} [c_{21}^{2}, c_{12}^{-1}], \quad c_{22}^{c_{12}} = [c_{12}, c_{21}^{-2}]\, c_{13}^{-c_{22}^{-2}}\, c_{22}\, c_{13}^{c_{31}^{-1}}.$$

The virtual braid group VB_n was introduced by L. Kauffman (1996).

VB_n is generated by the classical braid group $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ and the permutation group $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$. Generators $\rho_i, i = 1, \ldots, n-1$, satisfy the following relations:

$$\rho_i^2 = 1 \qquad \text{for} \quad i = 1, 2, \ldots, n-1, \qquad (1)$$

$$\rho_i \rho_j = \rho_j \rho_i \qquad \text{for} \quad |i-j| \geq 2, \qquad (2)$$

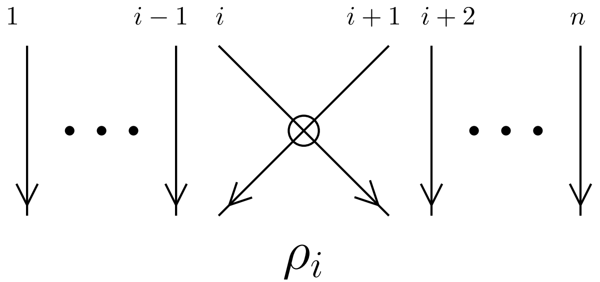
$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2 \ldots, n-2. \qquad (3)$$

Other defining relations of the group VB_n are mixed and they are as follows

$$\sigma_i \rho_j = \rho_j \sigma_i \qquad \text{for} \quad |i-j| \geq 2, \qquad (4)$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \qquad \text{for} \quad i = 1, 2, \ldots, n-2. \qquad (5)$$

The generators ρ_i have the following diagram



As in classical case there is a homomorphism

$$\varphi : VB_n \longrightarrow S_n, \quad \varphi(\sigma_i) = \varphi(\rho_i) = \rho_i, \quad i = 1, 2, \ldots, n-1.$$

Its kernel $\mathrm{Ker}(\varphi)$ is called the virtual pure braid group and is denoted by VP_n.

Define the following elements in VB_n:

$$\lambda_{i,i+1} = \rho_i \, \sigma_i^{-1}, \quad \lambda_{i+1,i} = \rho_i \, \lambda_{i,i+1} \, \rho_i = \sigma_i^{-1} \, \rho_i, \quad i = 1, 2, \ldots, n-1,$$

$$\lambda_{ij} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1},$$

$$\lambda_{ji} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i+1,i} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1}, \quad 1 \leq i < j-1 \leq n-1.$$

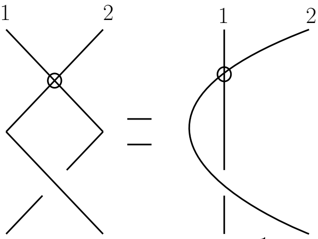
Theorem [V. B, 2004]

The group VP_n $(n \geq 2)$ admits a presentation with the generators λ_{ij}, $1 \leq i \neq j \leq n$, and the following relations:

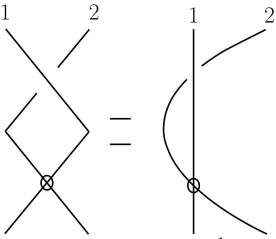
$$\lambda_{ij}\lambda_{kl} = \lambda_{kl}\lambda_{ij},$$
$$\lambda_{ki}\lambda_{kj}\lambda_{ij} = \lambda_{ij}\lambda_{kj}\lambda_{ki},$$

where distinct letters stand for distinct indices.

Note that $VP_2 = \langle \lambda_{12}, \lambda_{21} \rangle$ is 2-generated free group. The generators have geometric interpretation:



$$\lambda_{12} = \rho_1 \sigma_1^{-1}$$



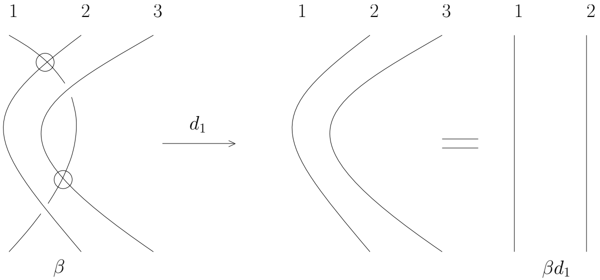
$$\lambda_{21} = \sigma_1^{-1} \rho_1$$

Let $VP_* = \{VP_n\}_{n \geq 1}$ be the set of virtual pure braid groups. Define the face map:

$$d_i : VP_n \longrightarrow VP_{n-1}, \quad i = 1, 2, \ldots, n,$$

what is the deleting of the ith strand.

Example:

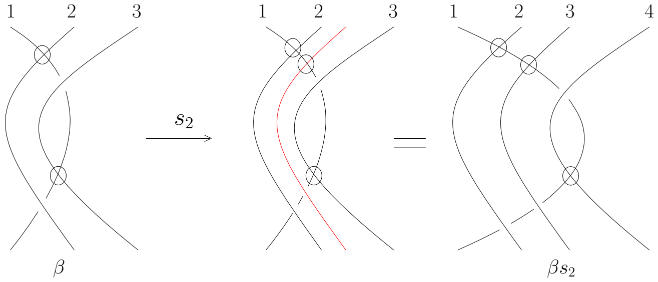


Define the degeneracy map:

$$s_i : VP_n \longrightarrow VP_{n+1}, \quad i = 1, 2, \ldots, n,$$

what is the doubling of the ith strand.

Example:



It is not difficult to see that we have the simplicial group

$$VAP_* \quad : \quad \cdots \rightleftarrows VAP_2 \rightleftarrows VAP_1 \rightleftarrows VAP_0,$$

where $VAP_n = VP_{n+1}$.

Proposition

VAP_* is contractible, i.e. $\pi_n(VAP_*) = 0$ for all $n \geq 1$.

Define a simplicial group $T_* = \{T_n\}_{n \geq 0}$ that is a simplitial subgroup of VP_* and is generated by λ_{12} and λ_{21}:

$$T_* \quad : \quad \cdots \rightleftarrows T_2 \rightleftarrows T_1 \rightleftarrows T_0,$$

where T_n, $n = 0, 1, \ldots$, is defined by the following manner

$$T_0 = \{e\}, \quad T_1 = VP_2, \quad T_{n+1} = \langle s_1(T_n), s_2(T_n), \ldots, s_{n+1}(T_n) \rangle.$$

If we let $a_{11} = \lambda_{12}$, $b_{11} = \lambda_{21}$, and

$$a_{ij} = s_n \ldots \widehat{s}_i \ldots s_1 a_{11}, \quad b_{ij} = s_n \ldots \widehat{s}_i \ldots s_1 b_{11}, \quad i + j = n + 1.$$

Then

$$T_n = \langle a_{kl}, b_{kl} : k + l = n + 1 \rangle, \quad n = 1, 2, \ldots$$

Problem

Find a set of defining relations for T_n, $n = 2, 3, \ldots$

Put $c_{ij} = b_{ij} a_{ij}$. It is not difficult to see that $c_{ij} \in P_{i+j}$.

Theorem [V. B., R. Mikhailov, V. V. Vershinin and J. Wu, 2016]

The group VP_3 is generated by elements

$$a_{11}, \quad c_{11}, \quad a_{21}, \quad a_{12}, \quad c_{21}, \quad c_{12}$$

and is defined by relations

$$[a_{21}, a_{12}] = [c_{21} a_{21}^{-1}, c_{12} a_{12}^{-1}] = 1,$$

$$a_{21}^{c_{11}} = a_{21}, \quad c_{21}^{c_{11}} = c_{21}, \quad a_{12}^{c_{11}} = a_{12}^{c_{12} c_{21}^{-1}}, \quad c_{12}^{c_{11}} = c_{12}^{c_{21}^{-1}},$$

i. e. $VP_3 = \langle T_2, c_{11} \rangle * \langle a_{11} \rangle$, $\langle T_2, c_{11} \rangle = T_2 \leftthreetimes \langle c_{11} \rangle$.

As a corollary of the previous theorem we have

Corollary

$T_2 = \langle a_{21}, a_{12}, b_{21}, b_{12} \rangle$ is defined by infinite set of relations

$$[a_{21}, a_{12}]^{c_{11}^k} = [b_{21}, b_{12}]^{c_{11}^k} = 1, \quad k \in \mathbb{Z},$$

that are equivalent to

$$[a_{21}^{c_{21}^k}, a_{12}^{c_{12}^k}] = [b_{21}^{c_{21}^k}, b_{12}^{c_{12}^k}] = 1, \quad k \in \mathbb{Z}.$$

Proposition [V. B., R. Mikhailov, J. Wu, 2018]

VP_4 is the HNN-extension with the base group

$$G_4 = \langle c_{11}, a_{21}, a_{12}, c_{21}, c_{12}, a_{31}, a_{22}, a_{13}, b_{31}, b_{22}, b_{13} \rangle$$

associated subgroups A and B and stable letter a_{11}, G_4 is defined by the following relations (here $\varepsilon = \pm 1$):
1) conjugations by c_{11}^{ε}

$$a_{21}^{c_{11}^{\varepsilon}} = a_{21}, \quad a_{12}^{c_{11}^{\varepsilon}} = a_{12}^{c_{12}^{\varepsilon} c_{21}^{-\varepsilon}}, \quad c_{21}^{c_{11}^{\varepsilon}} = c_{21}, \quad c_{12}^{c_{11}^{\varepsilon}} = c_{12}^{c_{21}^{-\varepsilon}},$$

$$a_{31}^{c_{11}^{\varepsilon}} = a_{31}, \quad a_{22}^{c_{11}^{\varepsilon}} = a_{22}, \quad a_{13}^{c_{11}^{\varepsilon}} = a_{13}^{c_{13}^{\varepsilon} c_{22}^{-\varepsilon}}, \quad b_{31}^{c_{11}^{\varepsilon}} = b_{31},$$

$$b_{22}^{c_{11}^{\varepsilon}} = b_{22}, \quad b_{13}^{c_{11}^{\varepsilon}} = b_{13}^{c_{13}^{\varepsilon} c_{22}^{-\varepsilon}},$$

2) conjugations by c_{21}^{ε}

$$a_{31}^{c_{21}^{\varepsilon}} = a_{31}, \quad a_{22}^{c_{21}^{\varepsilon}} = a_{22}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}}, \quad a_{13}^{c_{21}^{\varepsilon}} = a_{13}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}}, \quad b_{31}^{c_{21}^{\varepsilon}} = b_{31},$$

$$b_{22}^{c_{21}^{\varepsilon}} = b_{22}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}}, \quad b_{13}^{c_{21}^{\varepsilon}} = b_{13}^{c_{22}^{\varepsilon} c_{31}^{-\varepsilon}},$$

3) conjugations by c_{12}^{ε}

$$a_{31}^{c_{12}^{\varepsilon}} = a_{31}, \quad a_{13}^{c_{12}^{\varepsilon}} = a_{13}^{c_{13}^{\varepsilon} c_{31}^{-\varepsilon}}, \quad b_{31}^{c_{12}^{\varepsilon}} = b_{31}, \quad b_{13}^{c_{12}^{\varepsilon}} = b_{13}^{c_{13}^{\varepsilon} c_{31}^{-\varepsilon}},$$

$$a_{22}^{c_{12}^{-1}} = a_{13}^{c_{13}^{-1} c_{31}} a_{13}^{-c_{13}^{-1} c_{22}} a_{22} [c_{21}, c_{12}^{-1}], \quad a_{22}^{c_{12}} = [c_{12}, c_{21}^{-1}] a_{13}^{-c_{13} c_{22}^{-1}} a_{22} a_{13}^{c_{13} c_{31}^{-1}},$$

$$b_{22}^{c_{12}^{-1}} = b_{13}^{c_{13}^{-1} c_{31}} b_{22} b_{13}^{-c_{13}^{-1} c_{22}} [c_{21}, c_{12}^{-1}], \quad b_{22}^{c_{12}} = [c_{12}, c_{21}^{-1}] b_{22} b_{13}^{-c_{13} c_{22}^{-1}} b_{13}^{c_{13} c_{31}^{-1}}.$$

4) commutativity relations

$$[a_{21}, a_{12}] = [a_{31}, a_{22}] = [a_{31}, a_{13}] = [a_{22}, a_{13}] = 1,$$

$$[c_{21} a_{21}^{-1}, c_{12} a_{21}^{-1}] = [b_{31}, b_{22}] = [b_{31}, b_{13}] = [b_{22}, b_{13}] = 1.$$

Theorem [V. B., R. Mikhailov, J. Wu, 2018]

The group

$$T_3 = \langle a_{31}, \quad a_{22}, \quad a_{13}, \quad b_{31}, \quad b_{22}, \quad b_{13} \rangle$$

is defined by relations

$$[a_{31}, a_{22}^{c_{22}^m c_{31}^{-m}}] = [a_{31}, a_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = [a_{22}^{c_{22}^m c_{31}^{-m}}, a_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = 1,$$

$$[b_{31}, b_{22}^{c_{22}^m c_{31}^{-m}}] = [b_{31}, b_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = [b_{22}^{c_{22}^m c_{31}^{-m}}, b_{13}^{c_{13}^k c_{22}^{m-k} c_{31}^{-m}}] = 1.$$

where $k, m \in \mathbb{Z}$.

Let $n \geq 4$ and $\mathcal{R}^V(n)$ denote the defining relations of VP_n. By applying the homomorphism $s_t \colon VP_n \to VP_{n+1}$ to $\mathcal{R}^V(n)$, we have the following relations

$$s_t(\lambda_{ij})s_t(\lambda_{kl}) = s_t(\lambda_{kl})s_t(\lambda_{ij}),$$
$$s_t(\lambda_{ki})s_t(\lambda_{kj})s_t(\lambda_{ij}) = s_t(\lambda_{ij})s_t(\lambda_{kj})s_t(\lambda_{ki})$$

in VP_{n+1} for $1 \leq i, j, k, l \leq n$ with distinct letters standing for distinct indices, which is denoted as $s_t(\mathcal{R}^V(n))$.

Theorem [V. B., R. Mikhailov, J. Wu, 2018]

Let $n \geq 4$. Consider VP_n as a subgroup of VP_{n+1} by adding a trivial strand in the end. Then

$$\mathcal{R}^V(n) \cup \bigcup_{i=0}^{n-1} s_i(\mathcal{R}^V(n))$$

gives the full set of the defining relations for VP_{n+1}.

Corollary [V. B., R. Mikhailov, J. Wu, 2018]

The group T_n, $n \geq 2$ is generated by elements

$$a_{i,n+1-i}, \quad b_{i,n+1-i}, \quad i = 1, 2, \ldots, n,$$

and is defined by relations

$$[a_{i,n+1-i}, a_{j,n+1-j}]^{c_{11}^{k_1} c_{21}^{k_2} \ldots c_{n-1,1}^{k_{n-1}}},$$

$$[b_{i,n+1-i}, b_{j,n+1-j}]^{c_{11}^{k_1} c_{21}^{k_2} \ldots c_{n-1,1}^{k_{n-1}}},$$

where $1 \leq i \neq j \leq n$, $k_l \in \mathbb{Z}$.

Thank you!