On a certain sub-Riemannian geodesic flow on the Heisenberg group.

S.V. Agapov[1,2], M.R. Borchashvili[2]

1. Sobolev Institute of Mathematics, Novosibirsk, Russia
2. Novosibirsk State University, Novosibirsk, Russia

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Sub-Riemannian manifold

Let M^n be a smooth n-dimensional manifold. A smooth family

$$\Delta = \{\Delta(q) : \Delta(q) \subset T_q M^n \quad \forall q \in M^n, \quad dim\,\Delta(q) = k\}$$

of k-dimensional subspaces in the tangent spaces is called **completely nonholonomic distribution** if the vector fields from Δ and all their iterated commutators generate the whole tangent space at every point $q \in M^n$.

A triple $(M^n,\ \Delta,\ g)$, where g is the Riemannian metric on M^n is called **the sub-Riemannian manifold.**
A piece-wise smooth curve $\gamma : [0, t_0] \mapsto M^n$ is called **admissible** if $\dot{\gamma}(t) \in \Delta(\gamma(t))$ for all $t \in [0, t_0]$. The length of this curve is equal to

$$L = \int_0^{t_0} \sqrt{g(\dot{\gamma}(t), \gamma(t))}dt.$$

The distance $d(q_1, q_2)$ between two points q_1, q_2 on the manifold M^n is given by the following formula:

$$d(q_1, q_2) = \inf L(\gamma(t)),$$

where inf of the length is taken among all the admissible curves connecting q_0 and q_1. Such function $d(\cdot, \cdot)$ is called **sub-Riemannian metric** on M^n.

Optimal control problem and Pontryagin's maximum principle

Let $f_1, ..., f_k$ be tangent vector fields from the distribution Δ such that they generate Δ at every point $q \in M^n$. Consider the following optimal control problem

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q), \;\; u_i \in R, \;\; \int_0^{t_0} \sum_{i=1}^{k} u_i^2(t) dt \mapsto min, \quad q(0) = q_1, \; q(t_0) = q_2,$$

here t_0 is fixed. If a curve $q(t)$ is optimal, then there exist a covector λ and a constant $\nu \leq 0$ such that

$$\dot{q}(t) = \frac{\partial H}{\partial \lambda}(q(t), \lambda(t), u(t)), \quad \dot{\lambda}(t) = -\frac{\partial H}{\partial q}(q(t), \lambda(t), u(t)),$$

where

$$H(q, \lambda, u) =< \lambda, \sum_{i=1}^{k} u_i f_i(q) > + \nu \sum_{i=1}^{k} u_i^2,$$

and moreover

$$\frac{\partial H}{\partial u}\,(q(t), \lambda(t), u(t)) = 0.$$

Heisenberg group

The three-dimensional Heisenberg group H^3 is the group of matrixes $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with respect to multiplication, where $x, y, z \in R$. Its Lie algebra l is generated by the following elements:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The group H^3 acts on itself by **the left and right translations**: $l_g(h) = gh, \; r_g(h) = hg$. We shall denote the induced mappings of the tangent spaces as

$$l_{g^*} : TH^3_h \to TH^3_{gh}, \qquad r_{g^*} : TH^3_h \to TH^3_{hg}.$$

Let l_0 be the vector space generated by e_1 и e_2. **The left-invariant** distribution generated by l_0 consists of two-dimensional planes $L_x = l_{g^*} l_0$. Respectively, **the right-invariant** distribution consists of two-dimensional planes $R_x = r_{g^*} l_0$. The commutators of the vector fields e_1, e_2, e_3 look like:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0,$$

so the given distribution is completely nonholonomic.
A Riemannian metric is called **left-invariant** if it conserves for all left translations l_g. It is enough to define the metric at one point of the group (for example, in the unit), then it is possible to move it to other points via left translations.

Sub-Riemannian geodesic flow on the Heisenberg group

Theorem 1 (I.A. Taimanov, *Integrable geodesic flows of nonholonomic metrics*, **J. Dyn. Control Systems, 1997.)**

1) The geodesic flow of the sub-Riemannian metric on the Heisenberg group corresponding to the left-invariant Riemannian metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+x^2) & -x \\ 0 & -x & 1 \end{pmatrix}$$

and the right-invariant distribution is a Hamiltonian system with the following Hamiltonian function

$$H = \frac{1}{2(1+x^2+y^2)}((1+x^2)\lambda_1^2 + (1+y^2)\lambda_2^2 +$$
$$+y^2(1+x^2)\lambda_3^2 + 2xy\lambda_1\lambda_2 + 2y(1+x^2)\lambda_1\lambda_3 + 2xy^2\lambda_2\lambda_3)$$

and a standard symplectic structure.

2) This flow possesses three first integrals:

$$I_1 = H, \quad I_2 = \lambda_3(\sqrt{x^2+y^2} - y^2) + x\lambda_2 - y\lambda_1, \quad I_3 = \lambda_3.$$

3) Let us restrict this flow onto the level surface $\{I_3 = C_3 = const\}$ *and project the restriction of the flow onto the plane* (x, y). *The flow constructed in such a way is equivalent to a Hamiltonian system describing the motion of a charged particle on the two-dimensional plane with the following Riemannian metric*

$$ds^2 = (1+y^2)dx^2 - 2xydxdy + (1+x^2)dy^2 \tag{1}$$

in the constant magnetic field

$$\Omega = \lambda_3 dx \wedge dy. \tag{2}$$

Hamiltonian equations

In the polar coordinate system (r, ϕ), where $x = rcos\phi$, $y = rsin\phi$, metric (1) takes the form $ds^2 = dr^2 + (r^2 + r^4)d\phi^2$, the Hamiltonian is equal to

$$H(r, \phi, p_r, p_\phi) = \frac{1}{2}(p_r^2 + \frac{p_\phi^2}{r^2 + r^4}).$$

The symplectic structure is as follows:

$$\{r, p_r\} = \{\phi, p_\phi\} = 1, \quad \{p_r, p_\phi\} = C_3, \quad \{r, p_\phi\} = \{\phi, p_r\} = \{r, \phi\} = 0.$$

The Hamiltonian equations take the following form:

$$\dot{r} = \{r, H\} = p_r, \tag{3}$$

$$\dot{\phi} = \{\phi, H\} = \frac{p_\phi}{r^4 + r^2}, \tag{4}$$

$$\dot{p_r} = \{p_r, H\} = \frac{2r^3 + r}{(r^2 + r^4)^2} p_\phi^2 + C_3 \frac{p_\phi}{r^4 + r^2}, \tag{5}$$

$$\dot{p_\phi} = \{p_\phi, H\} = -C_3 p_r. \tag{6}$$

The first integrals of this system are

$$I_1 = H = \frac{1}{2}(p_r^2 + \frac{p_\phi^2}{r^4 + r^2}) = C_1, \quad I_2 = p_\phi + C_3 r = C_2.$$

The equations of motion read

$$\dot{r}^2 = 2C_1 - \frac{(C_2 - C_3 r)^2}{r^4 + r^2}, \quad \dot{\phi} = \frac{C_2 - C_3 r}{r^2(r^2 + 1)}. \tag{7}$$

Explicit formulae

Theorem 2 (A, B, *On a certain sub-Riemannian geodesic flow on the Heisenberg group*, **Sib. Mathem. Journ., 2017)**
1) If $C_2 = 0$, then the following formulae hold

$$t(r) = \pm \frac{1}{\sqrt{-2C_1}} E\left(\arcsin\left(r\sqrt{\frac{-2C_1}{2C_1 - C_3^2}}\right), \frac{2C_1 - C_3^2}{2C_1}\right),$$

$$\phi(r) = \phi_0 \mp \frac{C_3}{4\sqrt{2C_1 - C_3^2}} \Bigg(\ln\left(2C_1 - C_3^2\right) + 4\ln(r) -$$

$$-2\ln\left(4C_1\left(1+r^2\right) - C_3^2\left(2+r^2\right) + 2\sqrt{\left(2C_1 - C_3^2\right)\left(1+r^2\right)}\sqrt{-C_3^2 + 2C_1\left(1+r^2\right)}\right)\Bigg),$$

where $E(\psi, k^2)$ is the Legendre elliptic integral of the second kind.
In particular cases, for example, if $C_1 = \frac{1}{2}$, $C_3 = 1$, it is possible to express $t(r), \phi(r)$ in terms of elementary functions:

$$t(r) = \pm\sqrt{1+r^2} \pm ln\left|\frac{r}{2+2\sqrt{r^2+1}}\right|, \quad \phi(r) = \phi_0 \pm \frac{\sqrt{1+r^2}}{r}.$$

Explicit formulae

2) If $C_3 = 0$ (magnetic field vanishes), then the following formulae hold

$$t(r) = \pm \frac{\sqrt{C_1 + \sqrt{\varepsilon}}}{2C_1} \left(F(\psi, k^2) - E(\psi, k^2) \right),$$

$$\phi(r) = \pm \frac{i\sqrt{C_1 + \sqrt{\varepsilon}}}{C_2} \Pi(c^2, \theta, m^2),$$

where

$$\varepsilon = C_1(C_1 + 2C_2^2), \quad \psi = i \arcsin \left(\sqrt{2} \sqrt{\frac{C_1(1 + r^2)}{-C_1 + \sqrt{\varepsilon}}} \right),$$

$$k^2 = \frac{\sqrt{\varepsilon} - C_1}{C_2^2} - 1, \quad c^2 = \frac{C_1 + \sqrt{\varepsilon}}{C_2^2} + 2,$$

$$\theta = \arcsin \left(\sqrt{\frac{\sqrt{\varepsilon} - C_1(1 + 2r^2)}{2\sqrt{\varepsilon}}} \right), \quad m^2 = \frac{C_1 - \sqrt{\varepsilon}}{C_2^2} + 2,$$

here $F(\psi, k^2), E(\psi, k^2)$ and $\Pi(c^2, \theta, m^2)$ are the Legendre elliptic integrals of the first, the second and the third kinds accordingly.

Equations of motion

Denote

$$f(r) = r^4 + (1 - \frac{C_3^2}{2C_1})r^2 + \frac{C_2C_3}{C_1}r - \frac{C_2^2}{2C_1}.$$

Then (7) is equivalent to

$$\dot{r}^2 = 2C_1\frac{f(r)}{r^4 + r^2}.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be roots of the equation $f(r) = 0$, that is

$$f(r) = (r - \alpha_1)(r - \alpha_2)(r - \alpha_3)(r - \alpha_4).$$

By Vieta theorem the following relations hold:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0,$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = \frac{2C_1 - C_3^2}{2C_1},$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 = -\frac{C_2C_3}{C_1},$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = -\frac{C_2^2}{2C_1}.$$

This implies the following nontrivial relation on the roots:

$$\alpha_2^4(\alpha_3 - \alpha_4)^2 + 2\alpha_2^3(\alpha_3 - \alpha_4)^2(\alpha_3 + \alpha_4) + \alpha_3^2\alpha_4^2 - 2\alpha_2\alpha_3\alpha_4(\alpha_3 + \alpha_4)(2 + \alpha_3^2 + \alpha_4^2) +$$
$$+ \alpha_2^2(\alpha_3^4 - 2\alpha_3^3\alpha_4 - 2\alpha_3^2\alpha_4^2 + \alpha_4^4 - 2\alpha_3\alpha_4(2 + \alpha_4^2)) = 0.$$

Classification of trajectories

Theorem 3 (A, B, *On a certain sub-Riemannian geodesic flow on the Heisenberg group*, **Sib. Mathem. Journ., 2017)**
The Hamiltonian system (3)—(6) possesses the trajectories of four types.
Projections of trajectories of the 1st type onto the plane (r, ϕ) *lie outside the circle of the radius* r_0 *with center in the origin and have the following asymptotic:* $\phi \to const$ *while* $r \to +\infty$. *Here* r_0 *is the maximal real root of the equation*

$$f(r) = r^4 + (1 - \frac{C_3^2}{2C_1})r^2 + \frac{C_2 C_3}{C_1} r - \frac{C_2^2}{2C_1} = 0. \tag{8}$$

Projections of trajectories of the 2nd type onto the plane (r, ϕ) *lie inside the ring* $r_1 < r < r_2$, *where* r_1, r_2 *are real positive roots of the equation (8), and moreover* $f(r) > 0$ *while* $r_1 < r < r_2$.
Projections of trajectories of the 3rd type onto the plane (r, ϕ) *are circles given by the following formulae:*

$$r(t) = r(0) = r_3, \qquad \phi(t) = \phi(0) + \frac{C_2 - C_3 r_3}{r_3^4 + r_3^2} t,$$

where r_3 *is a real positive root of the equation (8) which also satisfies the following equation:*

$$r^3 - \frac{2C_2}{C_3} r^2 - \frac{C_2}{C_3} = 0.$$

Projections of trajectories of the 4th type onto the plane (r, ϕ) *are straight lines given by the following formulae:*

$$r(t) = r(0) \pm t\sqrt{2C_1}, \qquad \phi(t) = \phi(0).$$

Examples of trajectories

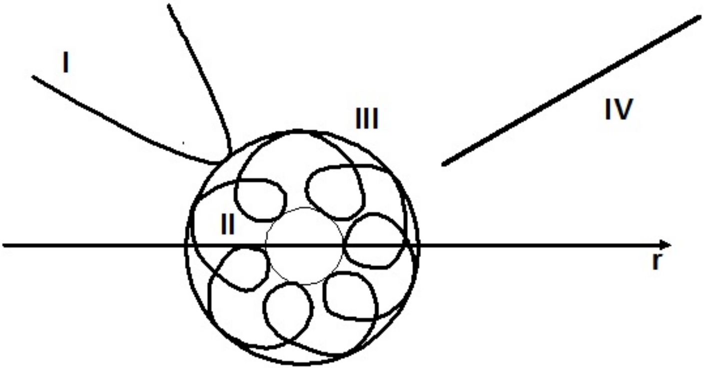

Рис.: Examples